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MASS TRANSFER THROUGH LAMINAR BOUNDARY LAYERS—3. SIMILAR SOLUTIONS OF THE *b*-EQUATION

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Abstract—As a preliminary to the presentation of general methods of mass transfer rate prediction, to follow in later papers, a study is made of the "similar" solutions of the differential equations governing the distribution of an arbitrary conserved property. The solutions available in the literature are collected. New solutions are presented valid for a two-parameter family of velocity distributions. It is shown that these suffice for the approximate prediction of the dimensionless conductance for a wide range of the variables: Prandtl/Schmidt number, Euler number, and driving force.

Résumé—A titre d'introduction à la présentation des méthodes générales d'évaluation du transport de matière qui doit suivre dans les prochains articles, on a fait ici l'étude des solutions "similaires" des équations différentielles régissant la distribution d'une propriété arbitraire qui se conserve. Les solutions fournies par la littérature ont été réunies. De nouvelles solutions, valables pour une famille de distributions de vitesses à deux paramètres sont présentées. On montre qu'elles suffisent pour une prévision approximative de l'évolution d'un grand nombre de variables: nombres de Prandtl, Schmidt, Euler et force dynamique.

Zusammenfassung—Als vorläufiges Ergebnis später folgender Arbeiten über allgemeine Methoden zur Bestimmung des übertragenen Stoffanteils, wird von „ähnlichen" Lösungen der Differentialgleichungen berichtet, die massgebend sind für die Verteilung einer beliebigen Erhaltungseigenschaft. Aus der Literatur verfügbare Lösungen sind zusammengestellt. Die Gültigkeit neuer Lösungen wird für eine zweiparamentrige Gruppe der Geschwindigkeitsverteilungen gezeigt. Diese Lösungen reichen aus, das dimensionslose Verhalten für einen grossen Bereich der Veränderlichen: Prandtlzahl, Schmidtzahl, Eulerzahl und der treibenden Kraft angenähert vorauszusagen.

Аннотация—Исходя из результатов предыдущей статьи, даётся предварительное рассмотрение общих методов расчёта скорости массообмена, основанных на «подобных» решениях дифференциальных уравнений при произвольных физических параметрах. Обобщаются также решения, имеющиеся в литературе. Предложенные новые решения справедливы при допущении двухпараметричности семейства профилей скорости в пограничном слое. Показано, что они пригодны для приближённого подсчёта безразмерного потока для большого числа переменных: чисел Прандтля, Шмидта, Эйлера и движущей силы (перепада давлений).

NOTATION

b = Dimensionless conserved property (equation 23), (—);
b'_0 = Gradient of b adjacent to the interface (equation 26), (—);
B = Driving force for mass transfer (equation 24), (—);
C = A constant (equation 6), (various);
c_f = Drag coefficient (section 5.2), (—);
Eu = Euler number (equation 44), (—);
f = Dimensionless stream function (equation 8), (—);
f_0 = Value of f at interface (equation 12), (—);
f''_0 = Dimensionless velocity gradient at interface (equation 48), (—);
f'''_0 = Dimensionless measure of $(\partial^2 u/\partial y^2)$ at interface (equation 48), (—);
$\mathbf{G}$ = Total mass flux vector (equation 1), (lb_m/ft^2h);

g = Mass-transfer conductance (equation 31); (lb_m/ft^2h);
g^* = Value of g for $B = 0$ (equation. 57), (lb_m/ft^2h);
I = Integral expression (equation 51), (—);
J = A coefficient (equation 52), (—);
K = A coefficient (equation 53), (—);
k = A constant, taken equal to zero in bulk of paper (equation 20), (—);
$\dot{m}''$ = Mass transfer rate per unit interface area (equation 4), (lb_m/ft^2h);
n = Exponent (equation 6), (—);
Nu = Nusselt number (equation 46), (—);
P = Any conserved property of the second class [1] (equation 1), (various);
Re = Reynolds number (equation 46), (—);
u = Velocity component parallel to interface (equation 5), (ft/h);
v = Velocity component normal to interface (equation 5), (ft/h);
W = Curvature parameter containing Δ_2 (equation 63), (—);
X = Curvature parameter containing Δ_4 (equation 60), (—);
Y = Parameter measuring rate of growth of Δ_4 (equation 61), (—);
Z = Parameter measuring rate of growth of Δ_2 (equation 62), (—);
x = Distance along interface in main-stream direction from start of boundary layer (equation 5), (ft);
y = Perpendicular distance from interface (equation 4), (ft);
β = A constant (equation 9), (—);
γ = Exchange coefficient ($\equiv$ diffusion coefficient $\times$ fluid density, or thermal conductivity $\div$ specific heat at constant pressure) (equation 1), (lb_m/ft h);
δ = "A" thickness of the velocity boundary layer (section 5.1), (ft);
δ_2 = Momentum thickness (ft);
Δ_2 = Convection thickness (equation 38), (ft);
Δ_4 = Conduction thickness (equation 33), (ft);
η = Dimensionless space co-ordinate (equation 8), (—);
μ = Dynamic viscosity of fluid (equation 32), (lb_m/ft h);
ν = Kinematic viscosity of fluid (equation 46), (ft^2/h);
ρ = Density of fluid (equation 5), (lb_m/ft^3);
σ = Prandtl or Schmidt number (equation 17), (—);
ϕ = A function of η (equation 18), (—);
ω = Variable proportional to η (equation 50), (—);

Subscripts
G = denotes fluid state in main-stream;
S = denotes fluid state adjacent interface;
T = denotes fluid state in transferred substance;
0 = denotes interface.

Also prime $'$ denotes differentiation with respect to η.

1. INTRODUCTION

1.1. *The standard problem of mass transfer theory*

THIS paper is the third in a series devoted to methods of predicting mass transfer rates through laminar boundary layers. The first two papers have been concerned with how the mass transfer through the phase interface affects the velocity distribution in the boundary layer; in these papers the rate of mass transfer was supposed known. We now turn to the main question: How is the mass transfer rate to be calculated?

A fairly general answer to the question has been given in a recent paper in this journal [1]: it is that the prediction of a steady mass transfer rate involves the solution of the equation:

$$\mathbf{G} \,.\, (\nabla P) - \nabla \{\gamma (\nabla P)\} = 0 \qquad (1)$$

with boundary conditions:

$$\text{in main-stream: } P = P_G \qquad (2)$$

$$\text{in fluid at interface: } P = P_S \qquad (3)$$

and

$$\dot{m}'' = \frac{[\gamma\, (\partial P/\partial y)]_S}{P_S - P_T} \qquad (4)$$

Here P is any conserved property, e.g. enthalpy (Btu/lb_m) or mass fraction of a chemically inert mixture component;
$\mathbf{G}$ is the local total mass flux vector (lb_m/ft^2h);
γ is the local exchange coefficient (diffusion coefficient times mixture den-

sity, or thermal conductivity divided by specific heat at constant pressure) (lb_m/ft h);

$\dot{m}''$ is the mass transfer rate, i.e. the component of $\boldsymbol{G}$ normal to the interface and directed towards the bulk of the fluid (lb_m/ft^2h);

y is the distance co-ordinate normal to the interface which is zero at the interface and positive within the fluid (ft);

P_G is the value of P within the main-stream (outside the boundary layer);

P_S is the value of P in the fluid immediately at the interface;

P_T is the value of P in the *transferred substance*, a concept which is discussed in [1].

Equation (1) does not stand alone, but must be solved simultaneously with the equations of momentum and continuity which govern the motion of the fluid; for $\boldsymbol{G}$ must be known as a function of position if (1) is to be solved for P. It is the latter equations which formed the subject of Papers 1 and 2 of the series [2, 3].

The present paper begins the discussion of equation (1) for the particular circumstances of the laminar boundary layer.

1.2. *An appreciation of the mathematical problem*

Equation (1) is a partial differential equation, the exact solution of which will ordinarily require extensive numerical work. Coupled with the fact that the momentum equation is of the same nature, this difficulty renders the obtaining of exact solutions an uneconomic task in most circumstances.

In Paper 1 it was shown how the difficulty with the momentum equation has been resolved for laminar boundary layers by the introduction of approximate methods which greatly reduce the labour of computation but still give acceptable accuracy in most circumstances. Corresponding approximate methods exist for equation (1) also.

The approximate method of Paper 1 involved the use of auxiliary functions, which were derived from a family of exact solutions of the equation of the velocity boundary layer; the latter are the so-called "similar" solutions, which were discussed in Paper 2 of the present series. It will therefore not be surprising to find that "similar" solutions of equation (1) perform an equally important role in the approximate methods of solving this equation.

The mathematical problem therefore has at least two important aspects; one of them concerns the obtaining of the family of similar solutions; the other concerns the use of the resulting functions in a general method of mass transfer prediction.

1.3. *Purpose and outline of present paper*

In Paper 1 we presented the approximate method for calculating the velocity boundary layer, and deferred until Paper 2 the detailed discussion of the functions derived from the similar solutions. In dealing with equation (1) however, the reverse procedure is more convenient. The present paper is therefore devoted to the similar solutions of equation (1) under laminar boundary layer conditions, and to the presentation of the important functional relationships. The use of these functions in the solution of the general problem of mass transfer prediction will be dealt with in subsequent papers.

It follows that the present paper is only of background interest to those readers who merely want to know *how* to calculate mass transfer rates. The paper is chiefly valuable to those who wish to know *why* and *under what circumstances* the procedures which are recommended in subsequent papers may be expected to work well.

In discussing the "similar solutions", we shall present a compilation of all the relevant data which we could find in previously published exact solutions. We shall then augment these data by Tables calculated by us.

In Section 2, we have presented the mathematics of the similar solutions at some length. Although the treatment is novel only in detail, we feel that the extended treatment is justified by the absence of any single appropriate publication to which we could refer. We have paid particular attention to the provision of formulae permitting transformation from one type of notation to another.

Section 3 contains the compilation of already published data, while Section 4 contains our own contributions. We provide graphs, e.g. Figs. 5, 6 and 7, which display some of the new data and permit comparison with the old.

Finally, Section 5 contains two re-presentations of the new solutions which will serve as the auxiliary functions in general approximate methods to be presented in later papers in the series.

2. THE MATHEMATICS OF THE SIMILAR SOLUTIONS

2.1. *The laminar-boundary-layer equation and boundary conditions*

We restrict consideration to two-dimensional* laminar boundary layers with uniform material properties. Using a rectangular co-ordinate system, the basic equation (1) now reduces to:

$$\rho u \frac{\partial P}{\partial x} + \rho v \frac{\partial P}{\partial y} - \gamma \frac{\partial^2 P}{\partial y^2} = 0 \qquad (5)$$

where ρ = fluid density, assumed uniform ($\mathrm{lb_m/ft^3}$);
x = distance along interface in flow direction (ft);
y = distance from interface into fluid (ft);
u = velocity component in x-direction (ft/h);
v = velocity component in y-direction (ft/h).

The "similar" velocity distributions. We have already established, in Paper 1, that a condition for the existence of "similar" velocity profiles is that the main-stream velocity u obeys the relation:

$$\frac{\mathrm{d}u_G}{\mathrm{d}x} = C\, u_G{}^n \qquad (6)$$

where C and n are constants.

Under these conditions, the same reference implies that:

$$u = u_G f' \qquad (7)$$

and

$$v = -\left(\frac{\nu}{\beta}\frac{\mathrm{d}u_G}{\mathrm{d}x}\right)^{1/2} \{\eta f'(\beta - 1) + f\} \qquad (8)$$

where $f' \equiv \mathrm{d}f/\mathrm{d}\eta$;

f is the dimensionless stream function which depends only on η;
η is the dimensionless distance co-ordinate, namely $y\{(\mathrm{d}u_G/\mathrm{d}x)/\nu\beta\}^{1/2}$;
ν is the kinematic viscosity of the fluid ($\mathrm{ft^2/h}$); and
β is a constant related to n by:

$$\beta = 1/(1 - n/2). \qquad (9)$$

Inserting these relations in equation (5), there results

$$u_G \frac{\mathrm{d}u_G}{\mathrm{d}x} \cdot \frac{\partial P}{\partial u_G} \cdot f' - \left(\frac{\nu}{\beta}\frac{\mathrm{d}u_G}{\mathrm{d}x}\right)^{1/2} \{\eta f'(\beta - 1) + f\} \frac{\partial P}{\partial y} - \frac{\gamma}{\rho}\frac{\partial^2 P}{\partial y^2} = 0. \qquad (10)$$

Meanwhile, the definitions imply:

$$\dot{m}'' = \rho v_S \qquad (11)$$

while, because η equals zero at the interface (S-state, equation (8)) implies:

$$v_S = -\left(\nu\beta \frac{\mathrm{d}u_G}{\mathrm{d}x}\right)^{1/2} \frac{f_0}{\beta} \qquad (12)$$

where f_0 is the value of f at $\eta = 0$.

These two relations enable the boundary condition (4) to be rewritten for a similar velocity layer as:

$$-\left(\nu\beta \frac{\mathrm{d}u_G}{\mathrm{d}x}\right)^{1/2} \frac{f_0}{\beta} = \frac{\gamma}{\rho}\, \frac{(\partial P/\partial y)_S}{(P_S - P_T)}. \qquad (13)$$

Now, since

$$\eta \equiv y \left\{\frac{\mathrm{d}u_G}{\mathrm{d}x} \Big/ \nu\beta\right\}^{1/2} \qquad (14)$$

we can write:

$$\left(\frac{\partial P}{\partial y}\right)_S = \left\{\frac{\mathrm{d}u_G}{\mathrm{d}x} \Big/ \nu\beta\right\}^{1/2} \left(\frac{\partial P}{\partial \eta}\right)_S. \qquad (15)$$

* The resulting solutions are relevant to axi-symmetrical flows also, by reason of the existence of the Mangler transformation, as explained in Paper 1 of the present series.

Combining (15) with (13), there results:

$$\left.\begin{aligned} -f_0 &= \frac{\gamma}{\rho\nu} \cdot \frac{(\partial P/\partial\eta)_S}{P_S - P_T} \\ &= \frac{1}{\sigma} \cdot \frac{(\partial P/\partial\eta)_S}{P_S - P_T} \end{aligned}\right\} \quad (16)$$

where $\sigma \equiv \rho\nu/\gamma$. (17)

Clearly σ stands for the Prandtl number if γ represents the thermal conductivity divided by specific heat; it stands for the Schmidt number if γ represents a diffusion coefficient multiplied by the mixture density. Which is appropriate depends of course on the nature of the conserved property represented by P [1].

Now it will be recalled from Paper 1 that, if the velocity profiles are to be similar, as has been postulated, f_0 has a constant value for a given boundary layer. We therefore deduce from equation (16) that, since σ is also a constant for a given fluid, $(\partial P/\partial\eta)_S$ must be proportional to $P_S - P_T$.

Conditions for similarity of P-profiles. So far we have not required that the P-profiles in the boundary layer should be geometrically similar to each other at successive stations downstream. We now introduce this restriction by making the postulate:

$$P - P_G = (P_T - P_S) \cdot \phi(\eta) \quad (18)$$

where

$\phi(\eta)$ stands for some function of η, and

$P_T - P_S$ may in general be a function of the stream velocity U_G.

Equation (18) satisfies the requirement deduced in the foregoing section from the boundary condition (16). It also satisfies the requirement, characteristic of all boundary layer flows, that the value of the conserved property in the main-stream, namely P_G, is independent of downstream distance.

Insertion of (18) in the differential equation (10) leads to:

$$\frac{\beta f' \phi u_G}{P_T - P_S} \frac{d}{du_G}(P_T - P_S) - f\phi' - \frac{1}{\sigma}\phi'' = 0 \quad (19)$$

where ϕ' stands for $d\phi/d\eta$ and ϕ'' for $d^2\phi/d\eta^2$.

Equation (19) is an equation with only one independent variable provided that the term

$$\frac{u_G}{P_T - P_S} \frac{d}{du_G}(P_T - P_S)$$

is a constant.

The condition for the validity of the similarity requirement (18) is thus:

$$\frac{u_G}{P_T - P_S} \frac{d}{du_G}(P_T - P_S) = \text{const.} = k \text{ (say)}. \quad (20)$$

The differential equation then becomes:

$$\phi'' + \sigma f\phi' - k\beta f' \phi'' = 0. \quad (21)$$

The boundary condition (16) meanwhile reduces to:

$$-f_0 = (\phi')_0/\sigma \quad (22)$$

suffix 0 again indicating evaluation at the interface, sometimes called the "wall", where $\eta = 0$.

Further restriction of the scope of the inquiry. Very few exact solutions have been derived for values of k different from zero [4]. Moreover, they have all been obtained for the case $B = 0$ (significance explained in [1] and below); they are thus of relatively minor importance in mass transfer studies.

The experimental circumstances under which the properties of the fluid adjacent to the interface and of the transferred substance, P_S and P_T, both vary so as to obey equation (20) will be very rare in practice. We have therefore restricted consideration to the case in which $P_T - P_S$ is independent of downstream distance (i.e. of u_G). Thus we take $k = 0$ in all further studies.

Inspection of equation (18), evaluated for the interface, reveals that, since P_G is a constant, *both* P_S and P_T must be constants. The physical interpretation of this varies according to the circumstances: if P is enthalpy, for example, we deduce that not only must the wall temperature remain uniform, but the heat transferred through the S control surface [1] per unit of mass transferred must also be uniform.

Since P_S and P_T are now to be considered as uniform, we can introduce the notation developed for this case in the paper just referred to. We then have:

$$b \equiv \frac{P - P_S}{P_S - P_T}\left(= \frac{P_G - P_S}{P_S - P_T} - \phi\right) \quad (23)$$

and

$$B \equiv b_G = \frac{P_G - P_S}{P_S - P_T}. \quad (24)$$

The differential equation (21) therefore reduces to

$$b'' + \sigma f b' = 0 \quad (25)$$

with boundary conditions:

$$\left.\begin{array}{l} \text{at} \quad \eta = 0: b = 0, b' = b_0' = -\sigma f_0 \\ \text{at} \quad \eta = \infty: b = B. \end{array}\right\} \quad (26)$$

The mathematical problem which we have to solve is thus specified by (25) and (26), wherein f is of course a function of η, obtainable from the solutions summarized in Paper 2, while the prime indicates differentiation with respect to η. We shall refer to equation (25) as the "similar" b-equation.

2.2. *Solving the "similar" b-equation*

The solution of equation (25) presents no difficulties; for, provided f is regarded as known, the equation is linear in b. The steps in the solving procedure, first devised by Pohlhausen [5] for the case $B = 0$, are as follows:

(i) By use of the integrating factor $\exp\{\int \sigma f \, d\eta\}$, equation (25) may be integrated; there results:

$$b' = \text{const.} \exp\{-\textstyle\int_0^\eta \sigma f \, d\eta\}. \quad (27)$$

Clearly the constant in (27) must equal b_0', the value of the b-gradient at the wall.

(ii) Making this insertion, and carrying out a further formal integration, we obtain

$$b = b_0' \textstyle\int_0^\eta \exp\{-\int_0^\eta \sigma f \, d\eta\} \, d\eta \quad (28)$$

in which the boundary condition: $b = 0$ at $\eta = 0$ has already been inserted.

After insertion of the boundary condition: $b = B$ at $\eta = \infty$, and further rearrangement, equation (28) yields expressions for the wall-gradient, namely:

$$b_0' = B / \textstyle\int_0^\infty \exp\{-\int_0^\eta \sigma f \, d\eta\} \, d\eta \quad (29)$$

and for the b-profile, namely:

$$\frac{b}{B} = \frac{\int_0^\eta \exp\{-\int_0^\eta \sigma f \, d\eta\} \, d\eta}{\int_0^\infty \exp\{-\int_0^\eta \sigma f \, d\eta\} \, d\eta}. \quad (30)$$

It is with the former quantity that we shall chiefly be concerned.

Discussion. It will be noted that (29) and (30) can be evaluated by quadrature once $f(\eta)$ and σ have been specified. This is a simple computational task.

The relation: $b_0' = -\sigma f_0$, has not yet been used; it is evidently not so much a boundary condition as a compatibility requirement between the b-profile and the f-profile from which it is deduced. The consequences of the requirement will appear below (Sections 2.4 and 4.4).

2.3. *Relations between the solution of the equation and quantities of physical significance*

Before discussing the evaluation of (29), we shall examine its relation to quantities which are used in more general mass transfer problems. These are: the mass-transfer conductance, g, and certain boundary-layer thicknesses. In addition, the opportunity will be taken to make connexion with quantities which conventionally appear in discussions of the similar solutions.

The mass-transfer conductance, g. Following Spalding [1] we define a conductance g by the "Ohm's Law" relation,

$$\dot{m}'' \equiv g \,.\, B. \quad (31)$$

The quantity g has the dimensions of lb_m/ft^2h and is equal to the mass velocity of the main-stream multiplied by the Stanton number.

From comparison of equation (31) with equation (11), (12), and (26), there results:

$$g/(\mu \, \rho \, du_G/dx)^{1/2} = \frac{1}{\beta^{1/2}\sigma} \cdot \frac{b_0'}{B}. \quad (32)$$

Comparison of equations (32) and (29) indicates how the conductance g can be deduced from the solution of the b-equation.

The conduction-thickness, Δ_4. It is often convenient to use the concept of "boundary-layer thickness" in solving mass transfer problems. As with the velocity boundary layer, many definitions are possible, and several are convenient. The first of these is Δ_4 (the notation is that of Smith and Spalding [17]), defined by:

$$\Delta_4 \equiv B/(\partial b/\partial y)_S. \tag{33}$$

Clearly Δ_4 is the distance at which a tangent to the b-profile at the interface intersects the line of maximum-b (Fig. 1).

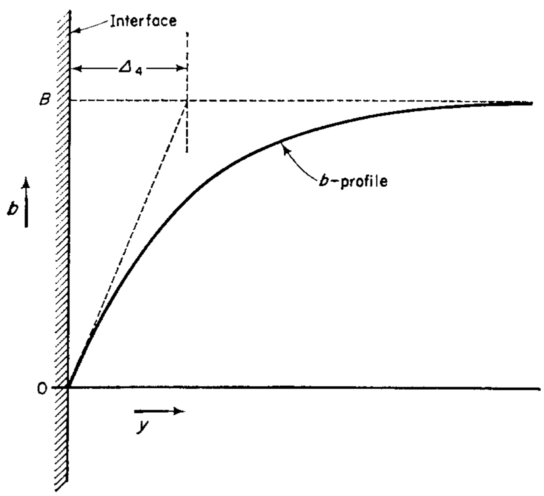

FIG. 1. Illustrating the relation of the "conduction thickness", Δ_4 to the variation of b normal to the wall.

To relate Δ_4 to other physical quantities and to the quantities derived from solution of the equation, we merely have to invoke the definition of η (see equation (14)). There results:

$$\Delta_4 \{(\mathrm{d}u_G/\mathrm{d}x)/\nu\}^{1/2} = \beta^{1/2} B/b'. \tag{34}$$

Three further relations involving Δ_4 are worth establishing. The first is obtained simply by squaring equation (34); there results an expression of a form which has proved to be useful in Paper 1, namely:

$$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \beta\left(\frac{B}{b_0'}\right)^2. \tag{35}$$

Since, for similar boundary layers, we also have

$$\frac{\mathrm{d}u_G}{\mathrm{d}x} = C\,u_G{}^n \tag{6}$$

it is possible to deduce from (35) that:

$$\frac{u_G}{\nu}\frac{\mathrm{d}\Delta_4^2}{\mathrm{d}x} = 2(1-\beta)\left(\frac{B}{b_0'}\right)^2. \tag{36}$$

The expression on the left-hand side of equation (36) is a measure of the rate of growth of the boundary-layer thickness; it will be found useful in approximate methods for predicting mass transfer rates.

The third relation involving Δ_4 is that which links it to the conductance, g. The definitions are readily seen to imply:

$$g\Delta_4/\gamma = 1. \tag{37}$$

This relation permits evaluation of g once Δ_4 has been deduced from a boundary-layer analysis.

The convection-thickness, Δ_2. Another convenient definition of a thickness of the boundary layer of b is:

$$\Delta_2 \equiv \int_0^\infty (u/u_G)(1 - b/B)\,\mathrm{d}y. \tag{38}$$

This definition implies that the quantity $\rho u_G B \Delta_2$ measures the convection flux along the boundary layer of the property b, measured below a base-value which prevails in the main-stream. Δ_2 is thus the analogue of the momentum thickness of the velocity boundary layer; we shall call it the "convection-thickness".

In relating Δ_2 to quantities appearing in the solution of the differential equation (25), we firstly introduce f' and η into (38), obtaining:

$$\Delta_2\{(\mathrm{d}u_G/\mathrm{d}x)/\nu\}^{1/2} = \beta^{1/2}\int_0^\infty f'(1 - b/B)\,\mathrm{d}\eta. \tag{39}$$

We then note that equation (25) can be integrated once formally to give:

$$[b']_0^\eta + \sigma\{[fb]_0^\eta - \int_0^\eta f'b\,\mathrm{d}\eta\} = 0. \tag{40}$$

Taking infinity as the upper limit, where $b = B$ and $b' = 0$, and noting that f_∞ can be written as $f_0 + \int_0^\infty f'\mathrm{d}\eta$, equation (40) becomes:

$$b_0' = \sigma B\{f_0 + \int_0^\infty f'(1 - b/B)\mathrm{d}\eta\}. \tag{41}$$

Finally we may eliminate the quadrature expression between equations (39) and (41) and replace σf_0 by $-b_0'$. There results:

$$\Delta_2\{(\mathrm{d}u_G/\mathrm{d}x)/\nu\}^{1/2} = \frac{\beta^{1/2}(1+B)}{\sigma}\frac{b_0'}{B} \tag{42}$$

which is the desired relation.

The final relation involving Δ_2 which we shall derive is that giving the ratio of boundary-layer thicknesses, Δ_2/Δ_4. This is obtained from (42) and (34), and is:

$$\frac{\Delta_2}{\Delta_4} = \frac{(1+B)}{\sigma}\left(\frac{b_0'}{B}\right)^2. \quad (43)$$

Relation to other quantities appearing in the literature. Solutions of the similar laminar-boundary-layer equations appearing in the literature are usually presented in terms of the distance x of the point in question from the start of the boundary layer. We here list, without comment, some relations which will enable quantities expressed in these terms to be translated into the present terms, and vice versa:

$$Eu = \beta/(2-\beta)$$
$$= 1/(1-n) \quad (44)$$

where

$Eu \equiv$ Euler number, i.e. the exponent in the relation between u_G, and x, namely

$$u_G \propto x^{Eu}. \quad (45)$$

$$Nu/\sqrt{Re} = \frac{x}{\Delta_4}\left(\frac{\nu}{xu_G}\right)^{1/2} = \frac{1}{(2-\beta)^{1/2}}\frac{b_0'}{B}$$
$$= [\{(2/\beta)-1\}\{(\Delta_4^2/\nu)(du_G/dx)\}]^{1/2} \quad (46)$$

$$\frac{\Delta}{x}\left(\frac{xu_G}{\nu}\right)^{1/2} = [\{(2/\beta)-1\}\{(\Delta^2/\nu)(du_G/dx)\}]^{+1/2} \quad (47)$$

where

$Nu \equiv$ Nusselt number;
$Re \equiv$ Reynolds number; and
Δ without suffix stands for either Δ_2 or Δ_4.

It should be noted that some of the expressions in (46) and (47) become indeterminate for $\beta = 0$, i.e. when u_G is independent of distance. It then becomes preferable to replace

$$\{(2/\beta)-1\}\{(\Delta^2/\nu)(du_G/dx)\}$$

by

$$\{(2-\beta)/2(1-\beta)\}\{(u_G/\nu)(d\Delta^2/dx)\}.$$

2.4. *Procedure for obtaining particular solutions*

It is clear that, whatever the form in which the solution is to be presented, the quadrature of (29) has to be evaluated. This equation therefore represents the key step in obtaining the solution. We now indicate the order in which the necessary steps may be taken.

We suppose that β and σ are given values and that it is desired to obtain values of the mass transfer conductance for various B values. The following procedure suffices:

(i) From the similar solutions of the velocity equation, seek out the one-parameter family of $f(\eta)$ functions corresponding to the β in question. The parameter is f_0.

(ii) For the particular σ in question, evaluate the quadrature of (29), thus obtaining b_0'/B as a function of f_0.

(iii) For each value of f_0, evaluate b_0' from the boundary condition (26).

(iv) Combining the results of steps (ii) and (iii), tabulate b_0' and b_0'/B for various B. Then use equation (32) or the appropriate one of the subsequent equations, for expressing g, Δ_4, Δ_2, etc., as functions of B.

3. SURVEY OF EXISTING SOLUTIONS OF THE "SIMILAR" *b*-EQUATIONS

3.1. *Tabulation of solutions*

Evaluations of the quadrature appearing in (29) have been made by numerous authors. Most of these, starting with Pohlhausen [5], have been concerned with the particular case in which the mass transfer is negligible (i.e. $f_0 = 0, B = 0$). More recently however, solutions have been published which are valid for non-zero B although that quantity has admittedly not often been calculated explicitly. We have collected all the published solutions which we were able to find and present quantities derived from them in the following tables.

The tabulated quantities are β, σ, B, b_0'/B, $(\Delta_4^2/\nu)(du_G/dx)$ or $(u_G/\nu)(d\Delta_4^2/dx)$, and four quantities termed X, Y, Z, and W; the latter quantities will be discussed below (Section 5.2).

Table 1. $B = 0$, *various* σ *and* β. These data have been derived from the solutions published in the pioneering papers of Pohlhausen [5], and of Eckert [6]. They are valid for heat transfer in the absence of mass transfer, or for mass transfer with very small driving forces. The data of Pohlhausen hold for $\beta = 0$, i.e. for the flat

Table 1. *Exact solutions of the b-equation:* $B = 0$

β	σ	$\frac{b_0}{B}$	$\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}$	X	Y	W	Z	References
−0·14	0·7	0·3698	−1·0238	−1·5807	4·9727	−0·8682	0·8238	
	0·8	0·3865	−0·9372	−1·5124	4·9778	−0·8302	0·8234	
	1·0	0·4160	−0·8090	−1·4052	4·9902	−0·7704	0·8224	Eckert [6]
	5·0	0·698	−0·2874	−0·8375	5·2820	−0·4463	0·7993	
	10·0	0·871	−0·1845	−0·6711	5·4369	−0·3525	0·7879	
0·0	0·1	0·1953	0	0	9·4561	0	0·5974	
	0·5	0·3664	0	0	7·1590	0	0·6866	
	0·6	0·3915	0	0	7·0416	0	0·6923	
	0·7	0·4139	0	0	6·9555	0	0·6966	
	0·8	0·4340	0	0	6·8925	0	0·6998	Pohlhausen [5]
	0·9	0·4526	0	0	6·8401	0	0·7024	with accuracy im-
	1·0	0·4696	0	0	6·8024	0	0·7044	proved by Merk [20]
	1·1	0·4856	0	0	6·7650	0	0·7063	
	7·0	0·9135	0	0	6·4681	0	0·7224	
	10·0	1·0298	0	0	6·4508	0	0·7233	
	15·0	1·1796	0	0	6·4372	0	0·7241	
0·2	0·7	0·4444	1·0127	0·6552	8·2179	0·2799	0·6409	
	0·8	0·4670	0·9171	0·6235	8·0933	0·2684	0·6458	
	1·0	0·5068	0·7787	0·5745	7·9154	0·2501	0·6530	
	5·0	0·898	0·2480	0·3242	7·1142	0·1489	0·6888	
	10·0	1·141	0·1536	0·2552	6·9363	0·1187	0·6976	
0·5	0·7	0·4705	2·2587	1·1455	9·3523	0·4588	0·6007	
	0·8	0·4951	2·0398	1·0886	9·1729	0·4402	0·6066	
	1·0	0·5390	1·7211	0·9999	8·8866	0·4108	0·6163	
	5·0	0·970	0·5314	0·5556	7·6235	0·2465	0·6654	
	10·0	1·240	0·3252	0·4346	7·2984	0·1971	0·6800	Eckert [6]
1·0	0·7	0·4959	4·0664	1·6360	10·6129	0·6151	0·5639	
	0·8	0·5225	3·6629	1·5527	10·3693	0·5906	0·5705	
	1·0	0·5704	3·0736	1·4223	9·9627	0·5519	0·5820	
	5·0	1·043	0·9193	0·7778	8·1475	0·3338	0·6436	
	10·0	1·344	0·5536	0·6036	7·6158	0·2679	0·6657	
1·6	0·7	0·5144	6·0467	2·0450	11·7331	0·7312	0·5363	
	0·8	0·5430	5·4265	1·9373	11·4001	0·7027	0·5441	
	1·0	0·5937	4·5393	1·7718	10·9023	0·6572	0·5564	
	5·0	1·098	1·3271	0·9581	8·6177	0·3997	0·6258	
	10·0	1·416	0·7980	0·7429	8·0357	0·3210	0·6481	

plate; those of Eckert hold for non-zero values of β.

Table 2. $\beta = 0$, *various* σ. The values contained in this table have mainly been deduced from the paper by Mickley, Ross, Squyers and Stewart [7], which represents the most extensive single source of exact solutions of the "similar" b-equations with non-zero B. Since $\beta = 0$ for these solutions, the underlying $f(\eta)$ relations are those for the flat plate in a uniform stream; we therefore tabulate $(u_G/\nu)\,(d\Delta_4^2/dx)$ rather than $(\Delta_4^2/\nu)\,(du_G/dx)$, since the latter quantity is zero throughout.

Only for $\sigma = 1$ have data other than those of

Table 2. Exact solutions of the b-equation: $\beta = 0$

σ	B	$\frac{b_0'}{B}$	$\frac{u_G}{\nu}\frac{d\Delta_4^2}{dx}$	X	Y	W	Z
0·6	5·171	0·08205	297·06	−8·618	81·73	−2·898	3·113
	2·023	0·1573	80·87	−3·372	30·64	−1·297	1·745
	0·9091	0·2333	36·73	−1·515	16·48	−0·6316	1·194
	0	0·3917	13·03	0	7·03	0	0·6927
	−0·3800	0·5582	6·419	0·633	3·828	0·3123	0·4585
	−0·5800	0·7314	3·738	0·967	2·372	0·4983	0·3247
	−0·6988	0·9107	2·411	1·165	1·594	0·6203	0·2407
	−0·8625	1·4757	0·9583	1·437	0·6465	0·8120	0·1165
	−0·9352	2·2687	0·3886	1·558	0·2824	0·9147	0·05708
0·7	6·774	0·0731	374·6	−9·671	134·7	−2·842	3·425
	2·448	0·1517	86·94	−3·500	39·9	−1·2597	1·8631
	1·054	0·2349	36·25	−1·506	18·84	−0·6085	1·2455
	0	0·4137	11·688	0	6·963	0	0·6960
	−0·4079	0·6067	5·434	0·583	3·479	0·2945	0·4489
	−0·6119	0·8089	3·056	0·874	2·046	0·4664	0·3106
	−0·7280	1·0199	1·923	1·040	1·324	0·5776	0·2267
	−0·8815	1·6846	0·7047	1·260	0·5072	0·7459	0·1054
	−0·9457	2·6170	0·2939	1·351	0·2146	0·8325	0·05021
0·8	8·739	0·0647	469·4	−10·92	222·0	−2·801	3·745
	2·915	0·1455	94·44	−3·644	51·58	−1·2301	1·982
	1·2034	0·2350	36·20	−1·504	21·50	−0·5898	1·2961
	0	0·4342	10·610	0	6·886	0	0·7001
	−0·4337	0·6522	4·70	0·5421	3·199	0·2794	0·4880
	−0·6398	0·8842	2·558	0·7997	1·790	0·4395	0·2970
	−0·7527	1·1274	1·573	0·9409	1·1203	0·5417	0·2137
	−0·8966	1·8929	0·5882	1·1207	0·4084	0·6908	0·09566
	−0·9537	2·9656	0·2274	1·1921	0·1687	0·7650	0·04452
0·9	11·147	0·0571	613·6	−12·39	363·9	−2·769	4·074
	3·433	0·1390	103·4	−3·810	66·54	−1·2063	2·103
	1·359	0·2343	36·44	−1·510	24·45	−0·5743	1·346
	0	0·4525	9·766	0	6·838	0	0·7026
	−0·4568	0·6965	4·123	0·5076	2·956	0·2666	0·4279
	−0·6645	0·9577	2·181	0·7383	1·5845	0·4162	0·2837
	−0·7744	1·2328	1·316	0·8604	1·1058	0·5100	0·2007
	−0·9085	2·1011	0·4530	1·0094	0·3359	0·6449	0·08755
	−0·9600	3·3142	0·1821	1·0667	0·1358	0·7082	0·03973
1·0	∞	0	∞	$-\infty$	∞	$-\infty$	∞
	178·70	0·004748	$8·8735 \times 10^4$	−178·70	$6·654 \times 10^4$	−11·376	17·159
	59·592	0·013647	$10·4711 \times 10^3$	−59·592	8054·1	−6·3295	9·6536
	31·829	0·024438	$3·4490 \times 10^3$	−31·829	2511·5	−4·4567	5·8951
	20·226	0·036709	$1·4841 \times 10^3$	−20·226	1113·1	−3·4205	5·3844
	14·077	0·050229	$7·9271 \times 10^2$	−14·077	594·54	−2·7457	4·4111
	10·358	0·064856	$4·7549 \times 10^2$	−10·358	356·61	−2·2639	3·7237
	7·9090	0·080462	$3·0893 \times 10^2$	−7·9090	231·69	−1·7304	2·9238
	6·1984	0·096966	$2·1271 \times 10^2$	−6·1984	159·53	−1·6126	2·8091
	4·9492	0·11430	$1·5309 \times 10^2$	−4·9492	114·815	−1·3798	2·4878
	4·0059	0·13239	$1·1411 \times 10^2$	−4·0059	85·582	−1·1866	2·2241
	3·2738	0·15119	98·750	−3·2738	65·621	−1·0234	2·0039
	2·6933	0·17065	68·678	−2·6933	51·508	−0·88326	1·8168
	2·2243	0·19074	54·976	−2·2243	41·229	−0·76183	1·6565
	1·8396	0·21140	44·751	−1·8396	33·565	−0·65536	1·5174
	1·5198	0·23262	36·959	−1·5198	27·720	−0·56123	1·3957
	1·2509	0·25437	30·911	−1·2509	23·182	−0·47740	1·2885
	1·0226	0·27660	26·141	−1·0226	19·606	−0·40232	1·1937
	0·82686	0·29931	22·325	−0·82686	16·744	−0·33451	1·1086
	0·65785	0·32246	19·235	−0·65785	14·426	−0·27314	1·0325
	0·51086	0·34603	16·703	−0·51086	12·527	−0·21729	0·96393
	0·38220	0·37001	14·609	−0·38220	10·956	−0·16627	0·90191
	0·26894	0·39437	12·859	−0·26894	9·6446	−0·11948	0·84559
	0·16872	0·41910	11·386	−0·16872	8·540	−0·076444	0·79429
	0·079600	0·44418	10·137	−0·079600	7·6028	−0·036735	0·74739
	0	0·46960	9·06935	0	6·8020	0	0·70439
	−0·071376	0·49534	8·1513	0·07136	6·1134	0·034070	0·66490
	−0·13562	0·52138	7·3573	0·13562	5·5180	0·065741	0·62850
	−0·19365	0·54772	6·6667	0·19365	5·0000	0·095244	0·59489
	−0·24623	0·57434	6·1999	0·24623	4·5458	0·12278	0·56379
	−0·29402	0·60123	5·5328	0·29402	4·1496	0·14853	0·53496
	−0·33758	0·62838	5·0650	0·33758	3·7988	0·17265	0·50818
	−0·37739	0·65585	4·6506	0·37739	3·4872	0·19528	0·48325
	−0·41386	0·68342	4·28205	0·41386	3·7386	0·21654	0·46002
	−0·44735	0·71129	3·9531	0·44735	2·9648	0·23655	0·43834
	−0·47817	0·73938	3·6584	0·47817	2·7438	0·25540	0·41807
	−0·53286	0·796195	3·1549	0·53286	2·3662	0·28998	0·38132
	−0·57973	0·85380	2·7436	0·57973	2·0577	0·32088	0·34892
	−0·62018	0·91213	2·4039	0·62018	1·8029	0·34863	0·32026
	−0·65531	0·97113	2·1207	0·65531	1·5905	0·37363	0·29479
	−0·68600	1·03077	1·88226	0·68600	1·4118	0·39624	0·27206

Table 2 (continued)

σ	B	$\frac{b_0'}{B}$	$\frac{u_G}{\nu}\frac{d\Delta_4^2}{dx}$	X	Y	W	Z
1·0	−0·71294	1·09100	1·68027	0·71294	1·2602	0·41674	0·25170
	−0·73671	1·15178	1·50762	0·73671	1·1307	0·43539	0·23340
	−0·75779	1·21306	1·35915	0·75779	1·0194	0·45240	0·21690
	−0·77654	1·27482	1·23064	0·77654	0·92298	0·46796	0·20200
	−0·79330	1·33703	1·11879	0·79330	0·83909	0·48223	0·18848
	−0·85512	1·65382	0·73123	0·85512	0·54842	0·53829	0·13680
	−0·89361	1·97824	0·51106	0·89361	0·383295	0·57661	0·10298
	−0·91899	2·3083	0·37535	0·91899	0·28152	0·60377	0·079833
	−0·94904	2·9803	0·22517	0·94904	0·16888	0·63851	0·051432
	−0·96529	3·6627	0·149085	0·96529	0·11181	0·65873	0·035532
	−0·97495	4·3516	0·105615	0·97495	0·079212	0·67147	0·025877
	−0·99039	7·1397	0·039235	0·99039	0·029426	0·69325	0·010092
1·1	17·713	0·04391	1037·2	−16·10	977·0	−2·917	5·084
	4·635	0·1259	126·25	−4·214	109·6	−1·200	2·408
	1·688	0·2304	37·684	−1·534	31·39	−0·5527	1·452
	0	0·4858	8·4751	0	6·758	0	0·7013
	−0·4971	0·7823	3·268	0·4519	4·873	0·2391	0·3991
	−0·7049	1·1034	1·643	0·6408	1·737	0·3663	0·2530
	−0·8089	1·4425	0·9612	0·7354	0·7774	0·4422	0·1725
	−0·9275	2·5154	0·3161	0·8432	0·1466	0·5441	0·07007
	−0·9693	4·0128	0·1242	0·8812	0·03611	0·5920	0·03108
1·4	34·063	0·02906	2368·0	−24·33	4292·0	−3·538	7·649
	6·956	0·1067	175·6	−4·969	228·8	−1·264	3·037
	2·145	0·2308	37·56	−1·532	39·76	−0·5299	1·632
2·0	116·86	0·01210	$1·366 \times 10^4$	−58·43	$8·490 \times 10^4$	−5·428	16·42
	14·300	0·07417	363·50	−7·150	974·0	−1·4668	4·708
	3·579	0·1976	51·240	−1·790	90·5	−0·5349	2·053
	0	0·5971	5·6100	0	6·617	0	0·6333
	−0·6264	1·1288	1·5763	0·3132	1·543	0·1528	0·2734
	−0·8177	1·7296	0·6686	0·4082	0·598	0·2135	0·1428
	−0·8957	2·3684	0·3565	0·4478	0·3021	0·2422	0·08463
	−0·9670	4·3876	0·10389	0·4835	0·0820	0·2726	0·02794
	−0·9872	7·1629	0·03898	0·4936	0·02990	0·2832	0·011045
5·0	3549·0	$0·9963 \times 10^{-4}$	$2014·8 \times 10^5$	−0·7098	$5·954 \times 10^9$	−419·7	314·8
	296·35	0·008948	2498·1	−59·27	$2·168 \times 10^4$	−2·588	19·47
	19·741	0·08955	249·41	−3·948	$2·430 \times 10^3$	−0·7201	5·674

Mickley *et al.* been used; here the source is the paper by Emmons and Leigh [8]. Although those authors were primarily concerned with the velocity boundary layer, the differential equation for the latter is identical with equation (25) for $\sigma = 1$ and $\beta = 0$; their data, which are more numerous than those of Mickley *et al.* for $\sigma = 1$, are therefore usable for the present purpose.

Table 3. $\sigma = 0{\cdot}7$, *various* β. These data have been taken from the following papers: Brown and Donoughe [9], Donoughe and Livingood [10], Livingood and Donoughe [11], Howe and Mersman [12].

In this table $(\Delta_4^2/\nu)\,(du_G/dx)$ has been tabulated in place of $(u_G/\nu)\,(d\Delta_4^2/dx)$. Of course, the one can easily be derived from the latter by way of equations (35) and (36).

3.2. *Graphical representation*

Most of the data contained in Tables 1, 2 and 3 are represented graphically in Figs. 2 and 3. In each case the ordinate represents the quantity $\sigma^{-1/3}\, b_0'/B$, which may be interpreted, in accordance with the foregoing equations, either as

$$\sigma^{-1/3}\,(2-\beta)^{1/2} Nu/\sqrt{Re}$$

or as

$$\sigma^{2/3}\beta^{1/2}/\,(\mu\;\rho du_G/dx)^{1/2}.$$

The $\sigma^{-1/3}$ is introduced into the ordinate quantity in order to bring the curves closer together; the reason that it does so will be apparent from the discussion which is to follow in Section 4.2. The quantity plotted in the abscissa is the mass transfer driving force B. Logarithmic paper is used in a manner permitting clear plotting.

Figure 2 is valid for $\beta = 0$ (the flat plate; uniform fluid stream velocity u_G) and for various values of σ. Fig. 3 is valid for a fixed value of σ, namely 0·7 which is a common value for gases, and shows the influence of β. The latter quantity, it will be recalled, measures the velocity gradient in the main-stream: $\beta = \frac{1}{2}$ corresponds to the stagnation point in axi-symmetrical flow;* while $\beta = 1$ corresponds to the stagnation point in a two-dimensional flow. The line marked $\sigma = 0{\cdot}7$ on Fig. 2 is of course identical with that marked $\beta = 0$ on Fig. 3.

It will be noted that relatively few exact solutions are available from which Fig. 3 can be plotted; in particular, no solutions are available for negative values of the driving force B for β values other than zero.

3.3. *Discussion*

Figures 2 and 3 contain almost all the published solutions of the b-equation, for uniform properties, in the Western literature, with the exception of those which lie on the lines $B = 0$. When it is borne in mind that, for complete coverage, we need a graph such as that of Fig. 2 for *every* value of β (or alternatively one like Fig. 3 for every value of σ), it is clear that much remains to be done. Even the figures which could be drawn are seriously incomplete: thus it appears that no one has obtained solutions for negative B (condensation, suction, absorption, and the like) for any case but that of zero velocity gradient.

Despite the paucity of the data, certain trends are quite clear. For example:

(i) The ordinate quantity falls as B increases, and rises as B decreases. Their relation depends on β and on σ.
(ii) The ordinate quantity tends to infinity as B tends to -1.
(iii) The ordinate quantity increases with increasing β.

* The appropriate transformation must be made before the $\beta = 0{\cdot}5$ curve can actually be used to evaluate g in axi-symmetrical stagnation-point flow. See Paper 1.

Table 3. Exact solutions of the b-equation: $\sigma = 0{\cdot}7$

β	B	$\frac{b_0'}{B}$	$\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}$	X	Y	W	Z	References
−0·1988	0	0·2955	−2·276	∞	0	−∞	∞	Brown and Donoughe [9]
−0·19	0	0·3278	−1·768	−6·75	2·565	−5·175	1·149	
−0·18	0	0·3411	−1·547	−4·115	3·400	−2·632	0·9964	
−0·16	0	0·3579	−1·249	−2·347	4·370	−1·377	0·8784	
−0·14	0	0·3698	−1·024	−1·581	4·975	−0·8681	0·8240	
−0·10	0	0·3873	−0·6665	−0·808	5·762	−0·4125	0·7651	
0	0	0·4139	0	0	6·963	0	0·6969	
0·6667	0	0·4806	2·886	1·355	9·830	0·5217	0·5861	
1·0	0	0·4958	4·017	1·636	10·620	0·6150	0·5638	
−0·0872	1·6300	0·1487	−4·060	∞	0	−∞	∞	Donoughe and Livingood [10]
0	1·0536	0·2347	0	−1·506	18·84	−0·6080	1·244	
0·6667	1·0120	0·2995	7·431	1·320	31·48	0·4083	0·9349	
1·0	1·1929	0·2934	11·62	1·813	40·6	0·5178	0·9399	
−0·007198	13·84	0·0356	−11·42	∞	0	−∞	∞	Livingood and Donoughe [11]
0	6·783	0·0730	0	−9·671	134·7	−2·843	3·425	
0·05	4·171	0·1216	6·441	−1·934	113·6	−0·5054	2·0263	
0·15	3·565	0·1489	11·77	−0·182	113·5	−0·04366	1·6828	
0·5	3·772	0·1607	25·80	1·332	156·0	0·2856	1·5332	
1·0	4·804	0·1457	47·11	2·206	256·9	0·4068	1·6030	
0·5	0	0·47049	2·259	1·1455	9·354	0·45871	0·60067	Howe and Mersman [12]
0·5	0·96426	0·29637	5·692	1·0202	28·381	0·32872	0·94935	
0·5	3·5510	0·16095	19·301	0·84342	132·23	0·19165	1·5512	

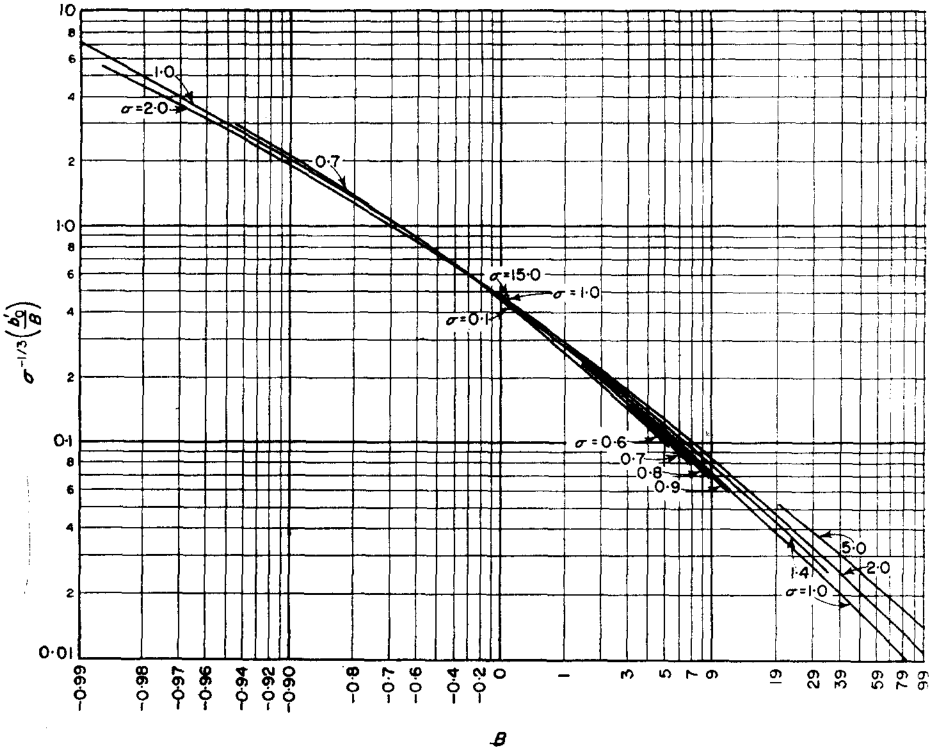

FIG. 2. Exact solutions to the b-equation for the flat plate ($\beta = 0$) from Tables 1 and 2. Parts of some curves have been omitted to retain clarity.

(iv) The ordinate quantity falls as σ decreases when B is positive, but rises slightly when B is negative.

We now consider ways in which the available data can be augmented.

4. NEW SOLUTIONS OF THE "SIMILAR" b-EQUATION

4.1. *Preliminary remarks*

It is clear from the foregoing discussion, together with inspection of equation (29), that the problem of the "similar" b-equation cannot be regarded as completely solved until (b_0/B) has been tabulated as a function of the three parameters: B, β and σ, over a sufficiently wide range of each of the variables for the tables to be completed by asymptotic formula. Although only quadratures are involved, the task is a formidable one. We have not attempted it.

Instead it has been our purpose to examine whether the solutions cannot be reduced to a two-parameter family, at least approximately; for such an approximation would reduce the computational task by one order of magnitude. The line of thought which we have pursued is that pioneered by Lighthill [13], and carried further by Spalding [14]; both these workers having been concerned with laminar boundary layers in the absence of mass transfer (case of $B = 0$).

We do not seek a two-parameter family of solutions merely so as to reduce labour: it is rather that we are here concerned with the similar solutions as means to an end, i.e. as the key to the prediction of mass transfer rates in more general, non-similar boundary layers. In this work a three-parameter family would be an embarrassment: there is already sufficient difficulty in handling one with two parameters, as will be seen in a later paper of the series.

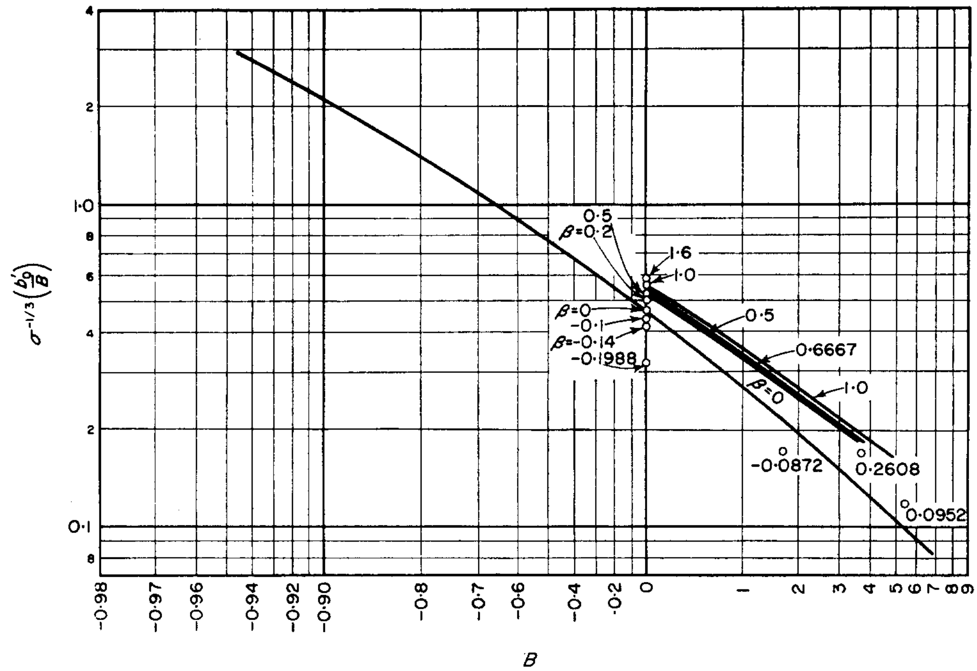

FIG. 3. Exact solutions to the b-equation for $\sigma = 0{\cdot}7$ and various values of β from Table 3.

Characteristic features of the solutions to be presented. In 1950, Lighthill showed how the earlier solution of Leveque [15] could be generalized so as to enable heat transfer rates (or mass transfer rates with B close to zero) to be calculated from knowledge of the velocity gradient in the y-direction close to the wall. The method is exact, provided that the b-boundary layer is much thinner than the velocity layer; this is the case with large σ.

Spalding [14] showed that the error in the Lighthill method can be greatly reduced if account is taken of the extent to which the b-layer reaches into the region where the velocity profile is appreciably curved. Thus, whereas Lighthill's analysis led to a single number, Spalding's led to a function of a single variable [Y (X) or Z (W) for $B = 0$ in the notation of Section 5.2].

The work now to be reported makes the next step: it includes the effect of non-zero B values. As a consequence we shall be concerned to examine the effect on the conductance of two parameters: "the curvature parameter" (X or W) which Spalding [14] found to be already of importance for zero values of B; and B itself.

Specifically, we shall be concerned with distributions of the stream function which can be expressed in the following form:

$$f = f_0 + \tfrac{1}{2} f_0'' \eta^2 + \tfrac{1}{6} f_0''' \eta^3. \tag{48}$$

The three analyses which have been mentioned can now be characterized as follows: *Lighthill* considered the case in which f_0 and f_0''' were zero; *Spalding* considered the case in which only f_0 was zero; in the *present paper*, all three coefficients of the expansion may be finite. Of course f_0' is always zero, since the fluid can be regarded as having zero u at the wall.

It should be understood that the solutions which follow are *exact* for the particular f-functions which are examined: the approximation only enters when the more complex f-functions of real velocity boundary layers are assumed to belong to this family.

4.2. *Mathematical discussion*

Transformation of the integration variable. On insertion of our particular expression for f, equation (48), in the integral which has to be evaluated, equation (29), we obtain the following relation:

$$B/b_0' = \int_0^\infty \exp\{-\sigma(f_0\eta + \tfrac{1}{6}f_0''\eta^3 + \tfrac{1}{24}f_0'''\eta^4)\}d\eta. \quad (49)$$

After introduction of a new variable of integration, ω, defined by:

$$\omega \equiv (\sigma f_0''/6)^{1/3}\eta \quad (50)$$

equation (49) reduces to:

$$(B/b_0')(\sigma f_0''/6)^{1/3} = \int_0^\infty \exp\{-J\omega - \omega^3 - K\omega^4\}d\omega \equiv I(J, K) \quad (51)$$

(say)

where

$$J \equiv \sigma f_0(6/\sigma f_0'')^{1/3} \quad (52)$$

$$K \equiv \frac{\sigma f_0'''}{24}\left(\frac{6}{\sigma f_0''}\right)^{4/3}. \quad (53)$$

This change of variable has the effect of leaving only two parameters, J and K, in the expression which has to be computed, namely the integral on the right-hand side of equation (51). Calling this integral I, it will now be our purpose to establish the function $I(J, K)$; with this known, equations (51), (52) and (53) will enable us to calculate (b_0'/B) for every set of values of f_0, f_0'', f_0''', and σ.

Some special cases. Where J and K are both equal to zero, the integral reduces to

$$\int_0^\infty \exp(-\omega^3)d\omega;$$

this has the value 0·89298, as was shown by Leveque [15] and Lighthill [13]. In this case, equation (51) shows that b_0'/B is equal to 0·616 $(\sigma f_0'')^{1/3}$. When J is positive and much larger than both unity and K, the integral reduces to

$$\int_0^\infty \exp(-J\omega)d\omega,$$

which has the value $1/J$; this arises when B tends to -1.

The determination of J, K and I. Equation (52) shows that J can be ascribed a particular number whenever f_0, f_0'' and σ are known. Now the relation between f_0 and f_0'' is known, for fixed values of the pressure-gradient parameter β, from solutions of the velocity equation (Papers 1 and 2). So J is fixed when σ, β and, say, f_0 are determined.

K may be evaluated in terms of velocity-boundary-layer parameters in a similar way. Here it is convenient to replace f_0''' by $-(f_0f_0''+\beta)$; the equivalence of these two quantities may be demonstrated directly by evaluation at $\eta = 0$ of the differential equation of the similar velocity boundary layer, namely (Paper 1):

$$f''' + ff'' + \beta(1 - f'^2) = 0. \quad (54)$$

Then equation (53) may be re-written; since $f' = 0$ at $\eta = 0$:

$$K = -\sigma\frac{(f_0f_0'' + \beta)}{24}\left(\frac{6}{\sigma f_0''}\right)^{4/3}. \quad (55)$$

Since I is a function of J and K, it now follows that it too can be evaluated if f_0, f_0'' and σ are prescribed.

4.3. *Computations and results*

A convergence difficulty. The quadrature of equation (51) may be evaluated numerically in a straightforward numerical manner provided that the quantity K is positive. When K is negative however, the integral ceases to be convergent. This mathematical fact may be expressed in physical terms as follows:

We are considering mass transfer into a boundary layer with a parabolic velocity profile. When the profile is concave upwards, as in curve (a) of Fig. 4, no difficulty arises; when the profile is concave downwards however, as in curve (b), the velocity must actually become negative at a certain distance from the boundary layer. The latter behaviour is physically unrealistic and is the cause of the non-convergence of the integral. To escape this difficulty, we have therefore modified the specification of the f-function for negative K, by requiring that the velocity profile (f') should remain quadratic up to the point of maximum velocity; thereafter the velocity is taken as independent of distance from the wall. This is shown by curve (c) in Fig. 4.

Expressing this modification symbolically, we

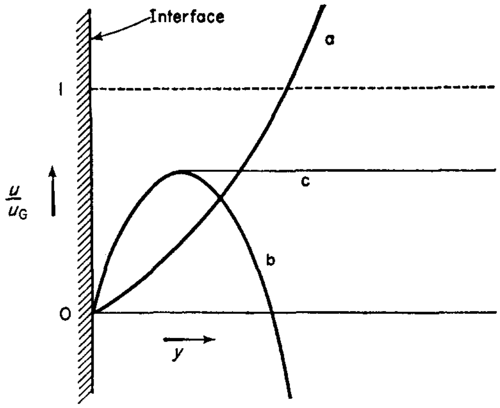

FIG. 4. Quadratic velocity profiles.

may write the new form of equation (51), valid for negative K as:

$$\left.\begin{aligned}&\left(\frac{B}{b_0'}\right)\left(\frac{\sigma f_0''}{6}\right)^{1/3}\\&= I \equiv \int_0^{-1/4K} \exp\{-J\omega - \omega^3 - K\omega^4\} d\omega\\&+ \int_{-1/4K}^{\infty} \exp\left\{\frac{1}{356K^3} + \left(\frac{1}{16K^2} - J\right)\omega\right.\\&\left.+ \frac{3}{8K}\omega^2\right\} d\omega.\end{aligned}\right\} \quad (56)$$

The expression on the right-hand side of (56) is convergent for all negative values of K.

Procedure for evaluation. The integral in equation (51) and the first integral in (56) were evaluated numerically, using Simpson's Rule. Generally an ω-interval of 0·1 was used, although a smaller interval was necessary when both J and K were large and positive. The resulting values of the integrals are believed to be correct to the fourth significant figure.

The second integral expression in (56) is expressible in closed form as:

$$\left(-\frac{2\pi K}{3}\right)^{1/2}\left\{\exp\left(\frac{1}{768K^3} + \frac{J}{12K} - \frac{2JK^3}{3}\right)\right\} \operatorname{erfc}\left\{\frac{1}{4\sqrt{6}(-K)^{3/2}} + J\left(\frac{-2K}{3}\right)^{1/2}\right\}.$$

This quantity was evaluated by reference to standard mathematical tables.

Results. The tabulated values of I are contained in Table 4. This table is divided into four parts: (a) J negative and K positive, (b) J positive and K positive, (c) J positive and K negative, (d) J and K both negative. Thus equation (51) was used for Tables 4(a) and 4(b); equation (56) was used for Tables 4(c) and 4(d).

4.4. *Deduction of new approximate solutions for real flows*

"Real" and "artificial" flows. Table 4, together with equations such as (26) and (51), enable *exact* values of b_0', B, etc. to be predicted for prescribed values of f_0, f_0'', β and σ, provided that the f profile has a form leading to (51) or (56).

Let us distinguish here between flows having the f-profiles just described and those obeying the "similar" laminar boundary-layer equations by calling the former "artificial" and the latter "real". Then Table 4 relates to artificial flows, while Tables 1, 2 and 3, containing the solutions obtained by other authors, relate to real flows.

The exact solutions for the artificial flow may serve as approximate solutions for the real flows if we assume that the only information about the velocity profile which influences b_0' is contained in the specifications of the three terms: f_0, f_0'' and f_0'''. We shall now make use of this assumption to derive new solutions for real flows with $\beta = 0$ (the flat plate) and $\beta = 1$ (the two-dimensional stagnation point). Since some exact solutions are available for these flows, (Tables 2 and 3), comparison with these will enable the accuracy of the approximation procedure to be discerned. Some solutions for other β-values are also derived.

Procedure. The solutions will be expressed in the same terms as those in Figs. 2 and 3, i.e. as plots of $\sigma^{-1/3} b_0'/B$ vs. B for various values of σ (see Section 3.2). Their deduction from the data in Table 3 proceeds as follows:

(i) For the β in question, choose a value of f_0.
(ii) Obtain the corresponding value of f_0'' from the exact solutions of the velocity equation tabulated in Paper 2.

(iii) Choose a value of σ.
(iv) Hence evaluate J from equation (52) and K from equation (53).
(v) Use the J and K values, in conjunction with Table 3, to give a value of I.
(vi) Hence obtain b_0'/B from equation (51), and so B from equation (26).
(vii) Finally calculate $\sigma^{-1/3}b'/B$.

Results. The approximate solutions obtained by this means are presented as full curves in Figs. 5 and 6, valid respectively for $\beta = 0$ and $\beta = 1$ and for various values of σ, and in Fig. 7, valid for $\sigma = 0{\cdot}7$ and various values of β. The points marked on the graph represent exact solutions taken from Tables 1, 2 and 3. The σ or β values for the curves and the points are indicated on the diagram. Only values of σ of 0·7 and above are considered.

The curves marked $\sigma = \infty$, though derived by the above procedure, may be regarded as exact. They correspond to the case in which the b-boundary layer is very much thinner than the velocity boundary layer, so that the velocity profile may be regarded as linear throughout the important region. This situation corresponds to a K-value of zero.

Table 4(a). *Values for I for various J and K obtained from evaluation of equation* (51)

J	K 3·0	2·5	2·0	1·5	1·0	0·8	0·6	0·4	0·2	0·0 K	J
−3·4	2·6644	2·9296	3·2867	3·7933							−3·4
−3·2	2·4019	2·6276	2·9293	3·3591							−3·2
−3·0	2·1702	2·3625	2·6177	2·9778	3·5377	3·8656	4·3126	4·9089			−3·0
−2·8	1·9653	2·1294	2·3455	2·6477	3·1117	3·3803	3·7325	4·2218			−2·8
−2·6	1·7838	1·9240	2·1073	2·3612	2·7463	2·9668	3·2547	3·6471	4·2370	5·2735	−2·6
										4·8348	−2·5
−2·4	1·6228	1·7426	1·8983	2·1120	2·4321	2·6135	2·8472	3·1647	3·6333	4·4378	−2·4
										4·0813	−2·3
−2·2	1·4796	1·5822	1·7146	1·8947	2·1613	2·3107	2·5016	2·7576	3·1263	3·7581	−2·2
										3·4669	−2·1
−2·0	1·3521	1·4401	1·5528	1·7048	1·9272	2·0506	2·2069	2·4148	2·7130	3·2025	−2·0
										2·9632	−1·9
−1·8	1·2384	1·3140	1·4100	1·6052	1·7243	1·8264	2·0879	2·1235	2·3622	2·7460	−1·8
										2·5488	−1·7
−1·6	1·1368	1·2017	1·2837	1·3925	1·5481	1·6326	1·7380	1·8754	2·0672	2·3691	−1·6
										2·2056	−1·5
−1·4	1·0459	1·1017	1·1711	1·2641	1·3944	1·4646	1·5515	1·6636	1·8173	2·0562	−1·4
										1·9168	−1·3
−1·2	0·9643	1·0124	1·0724	1·1508	1·2602	1·3186	1·3903	1·4820	1·6066	1·7953	−1·2
										1·6812	−1·1
−1·0	0·8910	0·9325	0·9840	1·0506	1·1427	1·1913	1·2506	1·3258	1·4267	1·5766	−1·0
										1·4806	−0·9
−0·8	0·8251	0·8609	0·9050	0·9618	1·0394	1·0800	1·1293	1·1910	1·2729	1·3923	−0·8
										1·3094	−0·7
−0·6	0·7656	0·7966	0·8346	0·8829	0·9485	0·9825	1·0233	1·0742	1·1408	1·2364	−0·6
											−0·5
−0·4	0·7119	0·7387	0·7714	0·8127	0·8682	0·8967	0·9307	0·9727	1·0271	1·1037	−0·4
											−0·3
−0·2	0·6633	0·6866	0·7147	0·7501	0·7978	0·8210	0·8494	0·8842	0·9286	0·9903	−0·2
											−0·1
0·0	0·6193	0·6395	0·6637	0·6940	0·7340	0·7541	0·7778	0·8066	0·8431	0·8930	0·0
J	K 3·0	2·5	2·0	1·5	1·0	0·8	0·6	0·4	0·2	0·0 K	J

Table 4(b). *Values for I for various J and K obtained from evaluation of equation* (51)

J	*K* 20	18	16	14	12	10	9	8	7	6	5	4	3	2	1	0 *K*	*J*
0·0						0·4866	0·4978	0·5104	0·5250	0·5419	0·5621	0·5871	0·6193	0·6637	0·7339	0·8930	0·0
0·1						0·4740	0·4846	0·4966	0·5102	0·5263	0·5453	0·5684	0·5984	0·6402	0·7050	0·8495	0·1
0·2						0·4618	0·4719	0·4832	0·4962	0·5112	0·5292	0·5511	0·5793	0·6178	0·6776	0·8090	0·2
0·4						0·4388	0·4479	0·4580	0·4696	0·4830	0·4988	0·5181	0·5411	0·5756	0·6276	0·7356	0·4
0·6						0·4174	0·4256	0·4347	0·4450	0·4570	0·4711	0·4882	0·5096	0·5388	0·5827	0·6729	0·6
0·8						0·3974	0·4047	0·4128	0·4219	0·4324	0·4446	0·4592	0·4795	0·5048	0·5425	0·6176	0·8
1·0						0·3790	0·3856	0·3929	0·4012	0·4107	0·4219	0·4352	0·4519	0·4739	0·5062	0·5689	1·0
1·4						0·3455	0·3509	0·3569	0·3635	0·3711	0·3800	0·3904	0·4033	0·4201	0·4442	0·4881	1·4
1·8						0·3163	0·3220	0·3255	0·3309	0·3370	0·3441	0·3523	0·3624	0·3752	0·3932	0·4249	1·8
2·2						0·2908	0·2944	0·2983	0·3027	0·3076	0·3132	0·3197	0·3276	0·3375	0·3510	0·3735	2·2
2·6					0·2632	0·2683	0·2712	0·2745	0·2780	0·2820	0·2864	0·2916	0·2978	0·3055	0·3157	0·3314	2·6
3·0					0·2442	0·2484	0·2509	0·2535	0·2564	0·2596	0·2632	0·2673	0·2722	0·2782	0·2860	0·2978	3·0
3·4					0·2273	0·2310	0·2327	0·2348	0·2372	0·2399	0·2429	0·2462	0·2501	0·2552	0·2607	0·2695	3·4
3·8					0·2122	0·2152	0·2168	0·2186	0·2205	0·2227	0·2250	0·2277	0·2308	0·2345	0·2391	0·2459	3·8
4·2					0·1987	0·2014	0·2026	0·2040	0·2056	0·2074	0·2093	0·2114	0·2139	0·2168	0·2204	0·2255	4·2
4·6				0·1848	0·1866	0·1887	0·1898	0·1910	0·1923	0·1938	0·1952	0·1971	0·1991	0·2014	0·2042	0·2080	4·6
5·0				0·1741	0·1757	0·1775	0·1783	0·1793	0·1804	0·1816	0·1828	0·1843	0·1859	0·1878	0·1900		5·0
5·4				0·1645	0·1658	0·1672	0·1681	0·1689	0·1698	0·1708	0·1717	0·1730	0·1743	0·1758			5·4
5·8				0·1559	0·1568	0·1582	0·1588	0·1595	0·1602	0·1610	0·1618	0·1628	0·1636	0·1651			5·8
6·2				0·1478	0·1487	0·1497	0·1503	0·1509	0·1515	0·1522	0·1528	0·1537	0·1546				6·2
6·6				0·1405	0·1413	0·1422	0·1426	0·1431	0·1436	0·1442	0·1447	0·1455					6·6
7·0	0·1320	0·1326	0·1332	0·1338	0·1345	0·1353	0·1356	0·1361	0·1365	0·1370	0·1374						7·0
7·4	0·1262	0·1267	0·1272	0·1277	0·1283	0·1290	0·1293										7·4
7·8	0·1207	0·1211	0·1216	0·1221	0·1226	0·1231	0·1234										7·8
8·2	0·1157	0·1160	0·1165	0·1169	0·1173												8·2
8·6	0·1111	0·1114	0·1118	0·1121	0·1125												8·6
9·0	0·1068	0·1071	0·1074	0·1077	0·1080												9·0
9·4	0·1028	0·1030	0·1033	0·1036	0·1039												9·4
J	*K* 20	18	16	14	12	10	9	8	7	6	5	4	3	2	1	0 *K*	*J*

Table 4(c). *Values for I for various J and K obtained from evaluation of equation* (56)

J	*K* 0·0	−0·1	−0·2	−0·4	−0·6	−0·8	−1·0	−1·4	−1·8	−2·2	−2·6	−3·0	−3·4	−3·8	−4·6 *K*	*J*
0·0	0·8930	0·9270	0·9730	1·1057	1·2533	1·3953	1·5287									0·0
0·1	0·8495	0·8793	0·9204	1·0361	0·1627	1·2826	1·3932	1·5901	1·7610							0·1
0·2	0·8090	0·8362	0·8731	0·9730	1·0829	1·1835	1·2759	1·4367	1·5730	1·6919	1·7887	1·8885				0·2
0·4	0·7356	0·7592	0·7862	0·8635	0·9449	1·0185	1·0839	1·1935	1·2822	1·3463	1·4193	1·4741				0·4
0·6	0·6729	0·6903	0·7126	0·7724	0·8337	0·8879	0·9350	1·0115	1·0711	1·1193	1·1593	1·1931	1·2221			0·6
0·8	0·6176	0·6316	0·6493	0·6957	0·7424	0·7830	0·8172	0·8719	0·9132	0·9457	0·9699	0·9937				0·8
1·0	0·5689	0·5803	0·5944	0·6307	0·6667	0·6973	0·7229	0·7626	0·7918	0·8144	0·8324	0·8470	0·8592	0·8696	0·8861	1·0
1·4	0·4881	0·4957	0·5048	0·5279	0·5492	0·5674	0·5821	0·6043	0·6201	0·6317	0·6410	0·6482	0·6536	0·6591	0·6668	1·4
1·8	0·4249	0·4295	0·4355	0·4502	0·4636	0·4748	0·4837	0·4969	0·5060	0·5126	0·5178	0·5218	0·5250	0·5277	0·5318	1·8
2·2	0·3735	0·3768	0·3813	0·3904	0·3991	0·4062	0·4119	0·4201	0·4257	0·4297	0·4328	0·4352	0·4371			2·2
2·6	0·3314	0·3343	0·3370	0·3424	0·3492	0·3540	0·3577	0·3629	0·3665	0·3702						2·6
3·0	0·2978	0·2995	0·3014	0·3057	0·3097	0·3128	0·3153	0·3189								3·0
3·4	0·2695	0·2707	0·2720	0·2749	0·2775	0·2798	0·2816	0·2832								3·4
3·8	0·2459	0·2465	0·2475	0·2496	0·2519											3·8
4·2	0·2255	0·2260	0·2267	0·2282												4·2
4·6	0·2080	0·2084														4·6
J	*K* 0·0	−0·1	−0·2	−0·4	−0·6	−0·8	−0·1	−1·4	−1·8	−2·2	−2·6	−3·0	−3·4	−3·8	−4·6 *K*	*J*

Table 4(d). Values for I for various J and K obtained from evaluation of equation (56)

J	K 0·0	−0·1	−0·2	−0·3	−0·4	−0·5	−0·6	−0·8	−1·0	−2·0	−3·0	−4·0	−5·0 K	J
−2·6	5·2735	6·2249	8·0749	11·8329										−2·6
−2·5	4·8348													
−2·4	4·4378	5·1558	6·5003	9·1101										−2·4
−2·3	4·1292													
−2·2	3·7581	4·3025	5·2850	6·9936										−2·2
−2·1	3·4669													
−2·0	3·2025	3·6166	4·3392	5·6319	7·5958									−2·0
−1·9	2·9632													
−1·8	2·7460	3·0625	3·5969	4·5185	5·8619	7·6755								−1·8
−1·7	2·5488													
−1·6	2·3691	2·6120	3·0095	3·6720	4·6016	5·8050	7·3142							−1·6
−1·5	2·2056													
−1·4	2·0562	2·2435	2·5409	3·0211	3·6715	4·4821	5·4580							−1·4
−1·3	1·9168													
−1·2	1·7953	1·9403	2·1640	2·5147	3·0191	3·5288	4·1616	5·7428						−1·2
−1·1	1·6812													
−1·0	1·5766	1·6893	1·8586	2·1168	2·4457	2·8294	3·2603	4·2599	5·4470					−1·0
−0·9	1·4806													
−0·8	1·3923	1·4804	1·6091	1·8006	2·0382	2·3076	2·6012	3·2514	3·9786					−0·8
−0·7	1·3094													
−0·6	1·2364	1·3053	1·4033	1·5469	1·7203	1·9118	2·1150	2·5470	3·0051	5·7143				−0·6
−0·5										4·6663	7·0791	9·9999		−0·5
−0·4	1·1037	1·1579	1·2337	1·3414	1·4690	1·6068	1·7497	2·0426	2·3389	3·8095	5·5295	7·3743		−0·4
−0·3										3·2560	4·4200	5·6176	6·8716	−0·3
−0·2	0·9903	1·0333	1·0924	1·1739	1·2681	1·3685	1·4704	1·6726	1·8690	2·7754	3·6071	4·4050	5·1899	−0·2
−0·1										2·3943	2·9984	3·5435	4·0508	−0·1
−0·0	0·8930	0·9270	0·9730	1·0343	1·1057	1·1796	1·2533	1·3953	1·5287	2·0880	2·5343	2·9152	3·2527	0·0
J	K 0·0	−0·1	−0·2	−0·3	−0·4	−0·5	−0·6	−0·8	−1·0	−2·0	−3·0	−4·0	−5·0 K	J

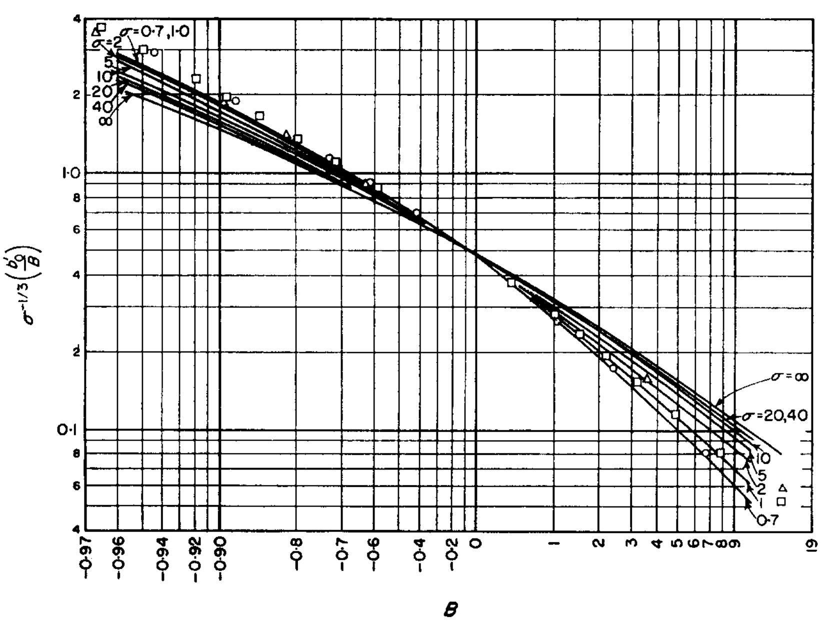

FIG. 5. $\sigma^{-1/3}$ (b_0'/B) vs. B for $\beta = 0$ and various σ deduced from Table 4. Points indicate exact solutions from Tables 1 and 2 (○ $\sigma = 0{\cdot}7$; □ $\sigma = 1{\cdot}0$; △ $\sigma = 2{\cdot}0$). Compare with Fig. 2.

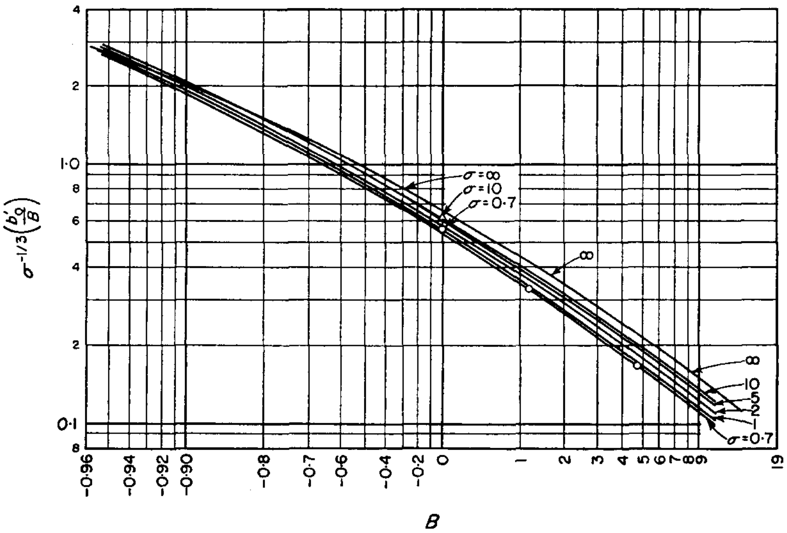

FIG. 6. $\sigma^{-1/3}$ (b_0'/B) vs. B for the two-dimensional stagnation point ($\beta = 1$) and various σ deduced from Table 4. Points mark exact solutions ($\sigma = 0{\cdot}7$ for ○, 10·0 for △).

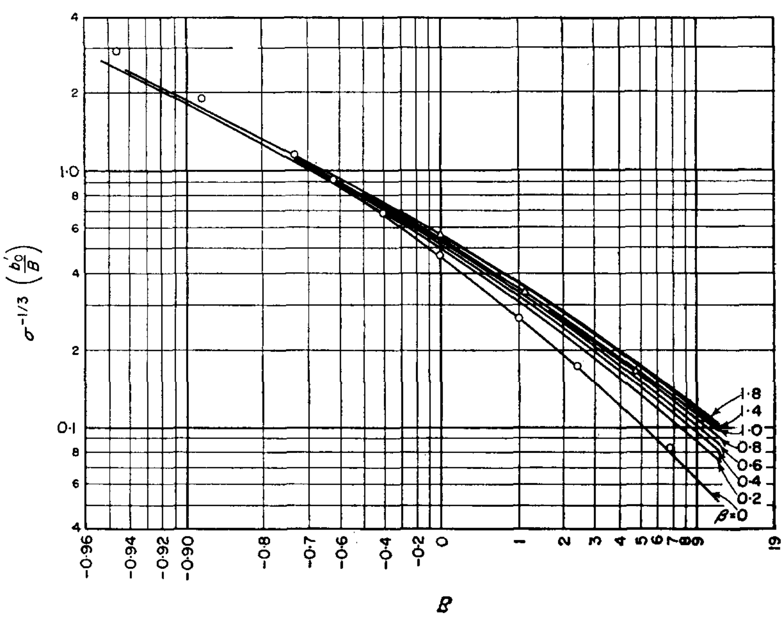

FIG. 7. $\sigma^{-1/3}$ (b_0'/B) vs. B for $\sigma = 0{\cdot}7$ and various β deduced from Table 4. Points mark exact solutions (○ $\beta = 0$; △ = $\beta = 1$). Compare with Fig. 3.

Discussion. Comparison of the points with the corresponding curves in Figs. 5 and 6 shows that the points lie on or close to the curves for large values of σ and moderate values of B, but tend to lie above them for the smaller values of σ, particularly as B tends to -1. However, the deviations are never large, not exceeding 10 per cent if B does not fall below $-0{\cdot}9$, and being within 5 per cent over most of the range. Fig. 7 reveals that, for $\sigma = 0{\cdot}7$, the discrepancy increases slightly as β increases.

Since very low values of B are rare in practice (the authors have only encountered values smaller than $-0{\cdot}9$ in studies of the burning of steel in oxygen jets), the approximate values of the ordinate quantity may be regarded as quite acceptable for most purposes.

Comparison of Figs. 2 and 3, on the one hand, with Figs. 5, 6 and 7 on the other, shows that a great deal of previously uncharted territory has been covered. Moreover, corresponding charts can be derived from the data in Table 3 for any other value of β or σ. It is clear that the neglect of higher terms than the second in the expansion of f' permits a large amount of (approximate) information to be obtained from quite a modest amount of computation.

The reason for the divergence between the approximate and the exact solutions for real flows is not hard to see. Our basic assumption requires that the b-boundary layer should be much thinner than the velocity boundary layer ($\Delta_4 \ll \delta_4$); this is valid for large σ, but not for small σ. Moreover the manner in which we have "horizontalized" the velocity profile (Fig. 4a) causes the assumed profile to differ appreciably from the real one when mass is being transferred out of the fluid at a high rate ($B \to -1$). It is possible that a different escape may be found from the convergence difficulty described in Section 4.3 that will involve less serious inaccuracy in this region.

One interesting feature of the solutions is that there is no tendency for the influence of B on g to disappear as σ becomes large; instead the curves take up an asymptotic shape which is independent of both β and σ. This means that even for diffusion in liquids, the mass transfer conductance is appreciably affected by B. The effect can be fairly well represented by the formula:

$$\frac{g}{g^*} = (1 + B)^{-0{\cdot}4} \tag{57}$$

where g^* is the value of g valid for $B = 0$ for the σ and β values which are in question. This formula is more exact than the logarithmic one used in the older literature (e.g. [16]) viz.

$$\frac{g}{g^*} = \frac{\ln (1 + B)}{B}. \tag{58}$$

5. FORMULATION OF THE "SIMILAR" SOLUTIONS AS AUXILIARY FUNCTIONS FOR MORE GENERAL METHODS

We have mentioned above that the "similar" solutions of the b-equation are chiefly of interest because they provide the auxiliary functions which are needed in solving "non-similar" problems; they act as bait enabling us to catch larger and more interesting fish.

The general methods for solving mass transfer problems will be described in later papers of the present series. In the present section, we merely re-arrange the solutions already presented in the form which the general methods require.

It will appear that the general methods fall into two classes (they are the classes designated I and II by Smith and Spalding [17]). Correspondingly the data from the similar solutions are required in two different forms. Section 5.1 provides data suitable for the Class I method of solving general mass transfer problems; section 5.2 provides data suitable for the Class II method.

5.1. *First re-presentation of the similar-solution data*

Although we do not intend to explain the Class I method at this point, it may be helpful to remark that it has much in common with that presented in Paper 1. There, it may be remembered, the relations between quantities such as (u_G/ν) $(d\delta^2/dx)$ and (δ^2/ν) (du_G/dx) proved to be important where δ represented "a" thickness of the velocity boundary layer; it should therefore occasion no surprise that we here inquire into the relations between (u_G/ν) $(d\Delta_4^2/dx)$ and (Δ_4^2/ν) (du_G/dx) which hold for the similar solutions.

We have seen, in Sections 2–3, how to relate these quantities to those obtained directly from the solution of the fundamental differential equation (25). Reflection concerning the number of independent parameters thereupon reveals that we may express the links between these quantities in the form:

$$\frac{u_G}{\nu}\frac{d\Delta_4^2}{dx} = F\left(\frac{\Delta_4^2}{\nu}\frac{du_G}{du},\ B,\ \sigma\right) \qquad (59)$$

where $F(\ldots)$ is a function the form of which is dictated by the similar solutions.

Figure 8 expresses the relation appearing in (59) in the form of a graph with B as the parameter; the graph holds for the σ-value of 0·7, which is typical of gaseous systems. This value has been chosen partly because of its practical importance and partly because it is the only σ-value for which any considerable number of exact solutions are available. However the

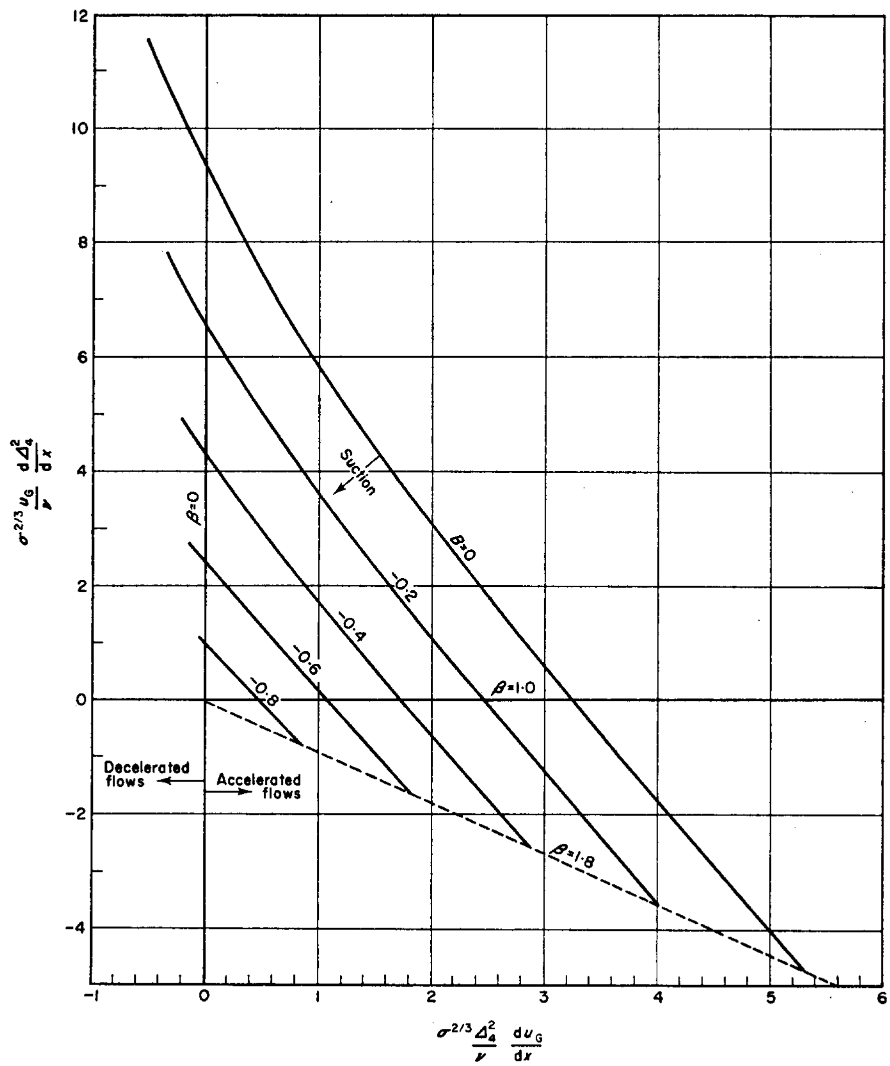

FIG. 8(a). $\sigma^{2/3}\frac{u_G}{\nu}\frac{d\Delta_4^2}{dx}$ vs. $\sigma^{2/3}\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}$ for $\sigma = 0{\cdot}7$ $-1 < B \leqslant 0$.

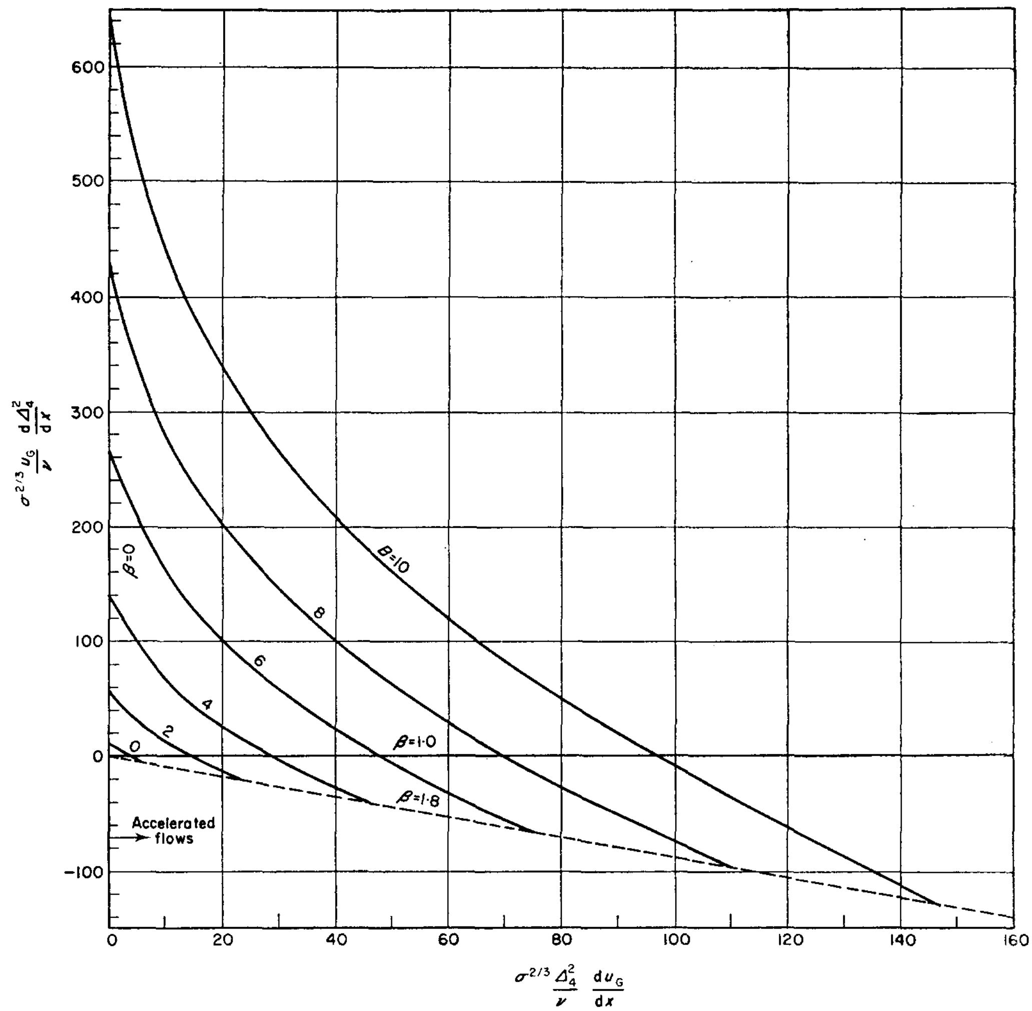

FIG. 8(b). $\sigma^{2/3}\frac{u_G}{\nu}\frac{d\Delta_4^2}{dx}$ vs. $\sigma^{2/3}\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}$ for $\sigma = 0{\cdot}7$ $0 \leqslant B \leqslant 10$.

ordinate and abscissa have been multiplied to $\sigma^{2/3}$, thus rendering the curves relatively insensitive to σ. Fig. 8 is presented in two parts: Fig. 8(a) exhibits negative values of B; Fig. 8(b) exhibits positive values of B. The curves have been derived from Table 4 by an obvious extension of the procedure described in Section 4.4, and then adjusted to pass through the points representing exact solutions extracted from Tables 1, 2 and 3. The curves thus represent the best estimates which we can make at the present time of the exact relation between the quantities in question. Were a greater number of exact solutions of the similar boundary-layer equations available, for real flows, we should not have had to make use of Table 4 at all.

Discussion. Without anticipating the later publication which is to deal comprehensively with the Class I method, only a few remarks can be made about the form of Fig. 8. The first is that it has roughly the same shape as one appearing in Paper 1 of the present series; there δ_2 took the place occupied by Δ_4 in Fig. 8, and the group $v_S\delta_2/\nu$ replaced B.

Secondly it should be mentioned that the line for $B = 0$ has already appeared in a publication by Smith and Spalding [17] dealing with heat transfer in laminar boundary layers; corresponding lines for other values of σ can be found in a report by Smith and Shah [18]. Finally, we remark that an early and rough version of Fig. 8, for positive values of B only, has been published by Spalding and Smith [19].

Of course it is possible to plot many graphs of the type shown in Fig. 8 by the use of Table 4 and of the exact solutions of the velocity equation; that for $\sigma = 0{\cdot}7$ is merely an example.

5.2. *Second re-presentation of the similar-solution data*

The Class II procedure which is to be described in a later paper in the series is an extension to finite mass transfer of that of Spalding [14]. The former publication was valid for heat transfer only (or mass transfer with $B \approx 0$); nevertheless we shall here represent the similar-solution data in the same co-ordinate system as was introduced there. Without proof or present explanation, we state that this implies the plotting of the quantity Y versus the quantity X for various values of B, and also the plotting of the quantity Z versus the quantity W for various values of B.

Definitions and connecting relations. The quantities X, Y, Z and W are related to other quantities already encountered by the following relations. In each case, that designated by (a) can be regarded as the definition, that designated (b) expresses the quantity in terms of quantities appearing in the similar solutions, (c) expresses it in terms of the quantities I, J and K, while (d) relates it to quantities appearing in the conventional heat transfer literature.

The relations are:

$$X \equiv \frac{\Delta_4\delta_4}{\nu}\frac{du_G}{dx} - \frac{v_S\Delta_4}{\nu} \tag{60a}$$

$$= \frac{-f_0'''}{f_0''}\frac{B}{b'} \tag{60b}$$

$$= -4IK \tag{60c}$$

$$= \frac{Eu}{(Nu/\sqrt{Re})(c_f\sqrt{Re}/2)} - \frac{(v_Sx/\nu)/\sqrt{Re}}{Nu/\sqrt{Re}} \tag{60d}$$

$$Y \equiv -\frac{\rho}{\gamma}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{d}{dx}\left\{\Delta_4^3\left(\frac{u_G}{\delta_4}\right)^{3/2}\right\} \tag{61a}$$

$$= \frac{3}{2}\sigma f_0''(B/b_0')^3 \tag{61b}$$

$$= 9I^3 \tag{61c}$$

$$= \frac{3}{4}\sigma\frac{(1+Eu)(c_f\sqrt{Re}/2)}{(Nu/\sqrt{Re})^3} \tag{61d}$$

$$Z \equiv \frac{\rho}{\gamma}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{d}{dx}(u_G\Delta_2)^{3/2} \tag{62a}$$

$$= \frac{3}{2}\frac{\sigma^{-1/2}(1+B)^{3/2}(b_0'/B)^{3/2}}{f_0''^{1/2}} \tag{62b}$$

$$= \left(\frac{3}{8}\right)^{1/2}\left(\frac{1}{I}-J\right)^{3/2} \tag{62c}$$

$$= \left(\frac{3}{2}\right)\left\{\frac{2}{\sigma(1+Eu)}\right\}^{1/2}\frac{\{(Nu/\sqrt{Re})+\sigma(v_Sx/\nu)/\sqrt{Re}\}^{3/2}}{(c_f\sqrt{Re}/2)^{1/2}} \tag{62d}$$

$$W \equiv \frac{\Delta_2^{1/2}\delta_4^{3/2}}{\nu}\frac{du_G}{dx} - \frac{\Delta_2^{1/2}\delta_4^{1/2}v_S}{\nu} \tag{63a}$$

$$= -\frac{(1+B)^{1/2}(b_0'/B)^{1/2}f_0'''}{\sigma^{1/2}f_0''^{3/2}} \tag{63b}$$

$$= \frac{-4}{6^{1/2}}K\left(\frac{1}{I}-J\right)^{1/2} \tag{63c}$$

$$= \left\{\frac{2}{\sigma(1+Eu)}\right\}^{1/2}\left\{\frac{Eu}{c_f\sqrt{Re}/2} - \frac{(v_Sx/\nu)}{\sqrt{Re}}\right\}\left\{\frac{Nu/\sqrt{Re}+\sigma(v_Sx/\nu)/\sqrt{Re}}{c_f\sqrt{Re}/2}\right\}^{1/2}. \tag{63d}$$

Once the definitions (a) are accepted, the other relations (b, c, d) follow from equations and definitions to be found elsewhere in this paper. For completeness we append the corresponding relations for the driving force B; they are:

$$B = \rho v_S\Delta_4/\gamma \tag{64a}$$

$$= -\sigma f_0\left(\frac{B}{b'_0}\right) \quad (64b)$$

$$= -IJ \quad (64c)$$

$$= \sigma\frac{(v_S x/\nu)/\sqrt{Re}}{Nu/\sqrt{Re}}. \quad (64d)$$

Graphical representation. Inspection of equations (60c), (61c), (62c), (63c) and (64c) shows that to every set of values of I, J, and K, there corresponds a set of values of X, Y, Z, W and B. Moreover I, J and K are linked via equation (51), which is expressed quantitatively by Table 4. It follows that the data of Table 4 can be cross-plotted in various ways.

The manner of cross-plotting which we choose is shown in Figs. 9 and 10. In the former, Y is plotted vs. X for various values of B; in the latter Z is plotted vs. W for various values of B.

Discussion. It would be inappropriate here to discuss the significance and utility of Figs. 9 and 10; these matters will be taken up in the later publications already referred to.

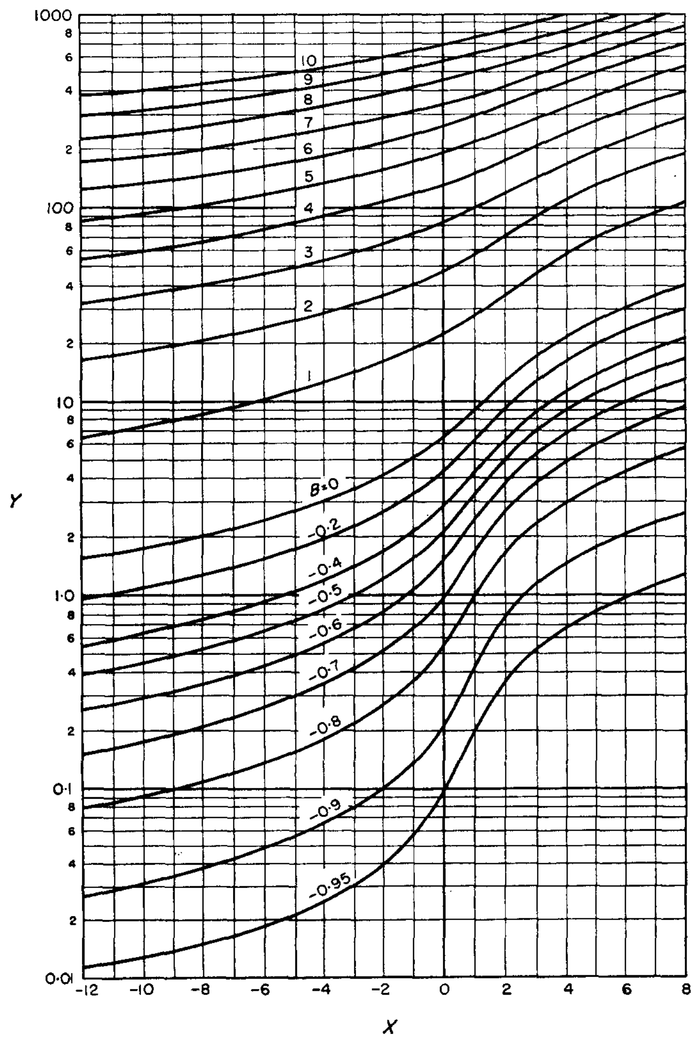

FIG. 9. X vs. Y with B as parameter.

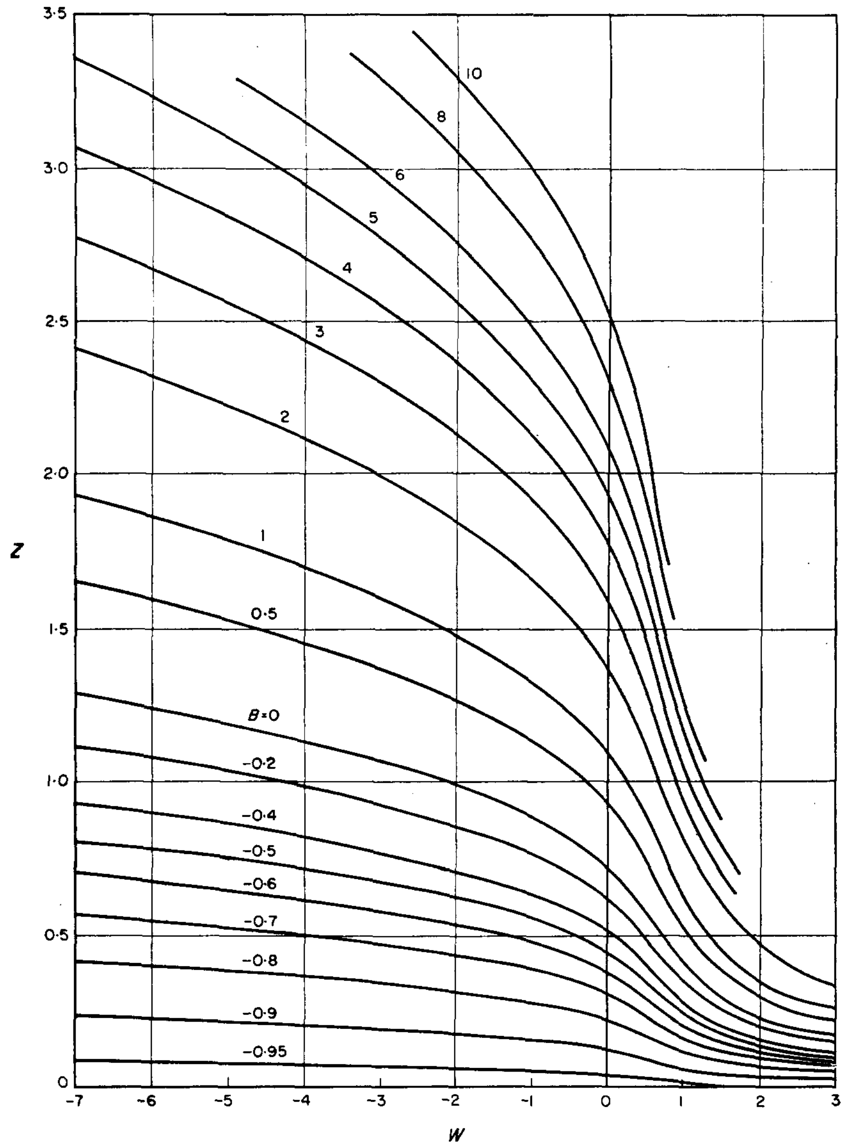

FIG. 10. Z vs. W with B as parameter.

6. CONCLUSIONS

(a) The differential equation for the distribution of a conserved property P in a laminar boundary layer has "similar" solutions provided: (i) that the velocity boundary layers are "similar", (ii) that the values of P in the fluid adjacent to the interface and in the transferred substance, P_S and P_T are related to the stream velocity u_G in accordance with equation (20).

(b) A survey has been made, and presented in Tables 1, 2 and 3 and in Figs. 2 and 3, of the relatively few exact solutions of the problem which are available in the literature. They are all for the case in which P_S and P_T are uniform.

(c) New solutions have been obtained by numerical quadrature for stream-function distributions characterized by three terms of a polynomial expansion (equation (48)). These have been used to generate new approximate solutions for real flows. Agreement with the exact solutions is mostly within a few per cent. The number of new solutions which can be generated from the tables greatly exceeds the number which have hitherto been available.

(d) The error in the approximate solutions is thought to stem in part from a modification to the three-term stream function distribution which has been introduced so as to prevent divergence of an integral. It is possible that less inaccurate modifications can be found.

(e) The new solutions have been displayed, in Figs. 8, 9 and 10, in ways which permit their use in the solution of non-similar mass transfer problems.

(f) The effect of mass transfer on the value of the conductance can be approximately represented, over a fairly wide range of conditions, by the equation: $g/g^* = (1 + B)^{-0 \cdot 4}$.

7. APPEAL

We here repeat the appeal made at the end of Paper 2 of the series. Conscious that our survey of the world's mass transfer literature has been incomplete, we ask readers to tell us of exact constant-property solutions which we have missed. We shall be particularly grateful to learn of current computational programmes in this field.

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